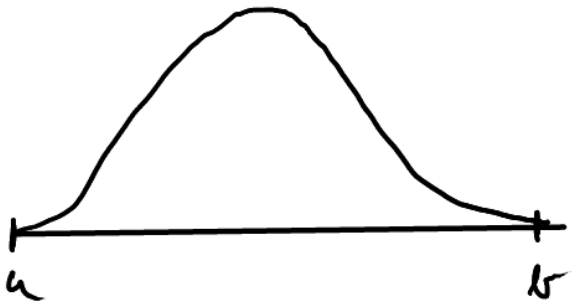
Grid generation 6.1 – 6.3

1D-case $\Omega = [a, b]$

Uniform grid, Choose $N \in \mathbb{N}$, $h = \frac{b-a}{N}$

$x_i = a + ih \quad i = 0, \dots, N$



Variable step size $h_i = x_{i+1} - x_i$

Given a spacing function

$H(x) > 0$ on $[a, b]$

$H : [a, b] \to \mathbb{R}$

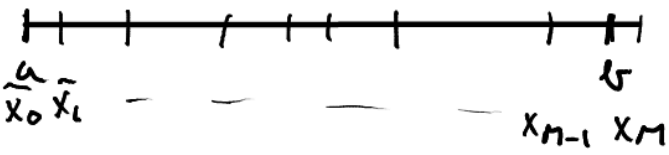
Want $h_i \approx H(x_i)$

Algorithm

$\tilde{x}_0 = a$

for $i = 1, 2, \dots$

$\tilde{x}_i = \tilde{x}_{i-1} + H(\tilde{x}_{i-1})$



stop when $x_i > b$ and $x_{i-1} < b$

If $x_M - b < b - x_{M-1}$ then $N = M$ otherwise $N = M - 1$

(i.e choose point closest to b)

let $k = \frac{b - x_{N-1}}{x_N - x_{N-1}}$

for $i = 1, \dots, N$

$x_i = \tilde{x}_i + k\, H(\tilde{x}_i)$

end

(Move them slightly so that the last one hits b exactly)

Boundary value problem method (BVP

Given $\Omega = [a, b]$

Want to distribute $N+1$ gridpoints in Ω,

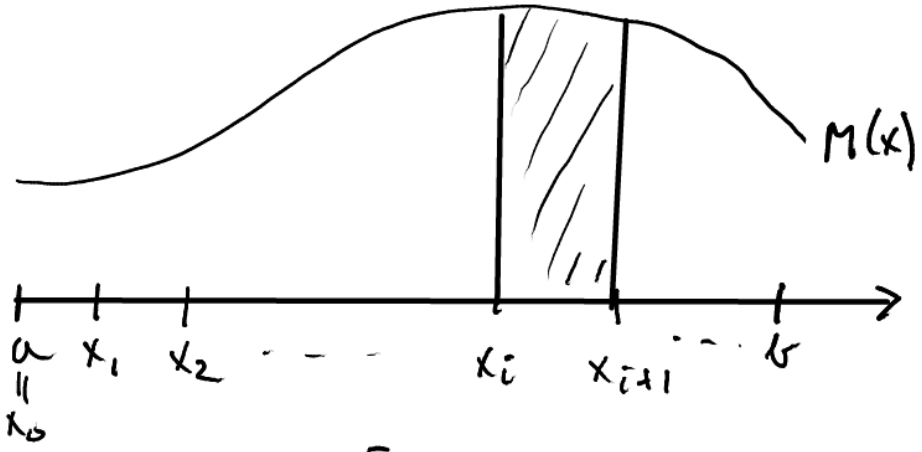
$x_0 = a \quad x_N = b$

Define a monitor function $M : \Omega \to \mathbb{R}$

Example: Given u or some approximation

$$M = \sqrt{1 + u_x^2}$$

$\int_\Omega M \, d\lambda$ = length of the curve $(x, u(x))$ in $\mathbb{R}^2$.



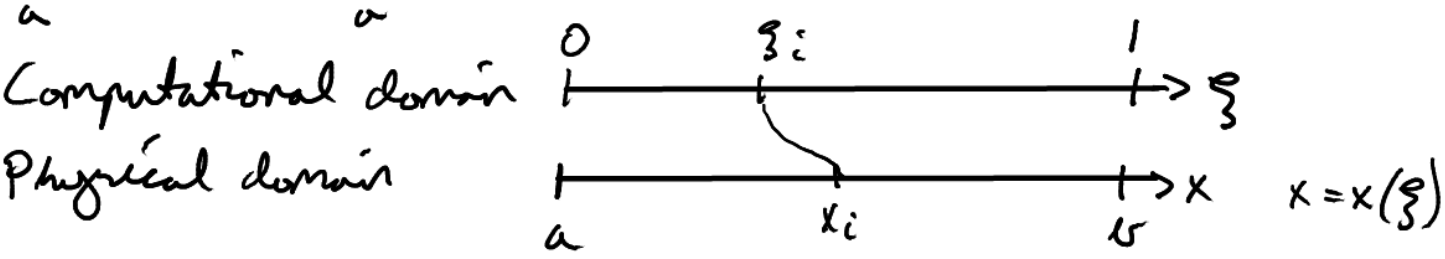
Equidistribution principle:

Find x_i such that

$$\int_{x_i}^{x_{i+1}} M(x) = \frac{1}{N} \int_a^b M(x) \, dx$$

Non-discrete version

$$\int_a^x M(\tau) \, d\tau = \xi \int_a^b M(x) \, dx \qquad \xi \in [0, 1]$$



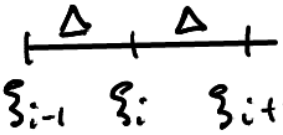
Uniform grid on computational domain

Remember $x = x(\xi)$. Differentiate twice w.r.t ξ

$$\frac{\partial}{\partial \xi}\left(M(x(\xi)) \frac{\partial x}{\partial \xi} \right) = 0 \qquad x(0) = a, \quad x(1) = b \qquad (BVP)$$

Solve (BVP) with some finite difference method.

$\Delta = \frac{1}{N}$

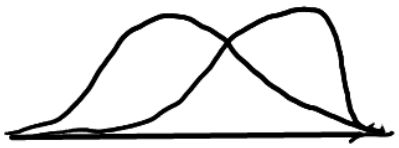


$$\frac{1}{\Delta}\left[\frac{M_{i+1} + M_{i-1}}{2} \frac{x_{i+1} - x_i}{\Delta} - \frac{M_i + M_{i-1}}{2} \frac{x_i - x_{i-1}}{\Delta} \right] = 0 \qquad i = 1, \dots, N-1$$

$M_i = M(x_i)$

Moving mesh For example

$$\frac{\partial u}{\partial t} = f(u) \qquad \frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} \frac{\partial}{\partial x}(u^2) \qquad u(0) = u(1) = 0$$

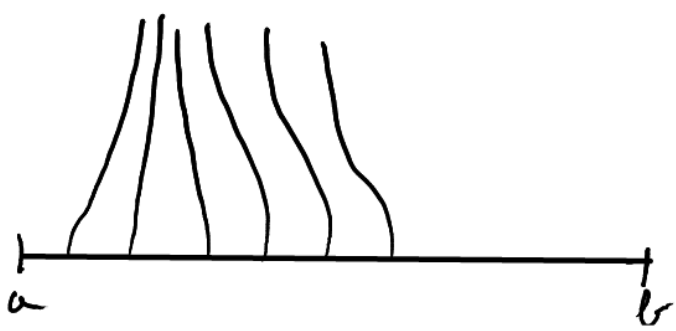


$u(x,t)$ x_i are equidistributed.

Trick to solve (BVP)

Construct a parabolic equation which has $x(\xi)$ as a stationary solution. $x = x(\xi, t)$

$$\frac{\partial x}{\partial t} = \frac{1}{\tau}\left(M x_\xi \right)_\xi \qquad \tau > 0$$



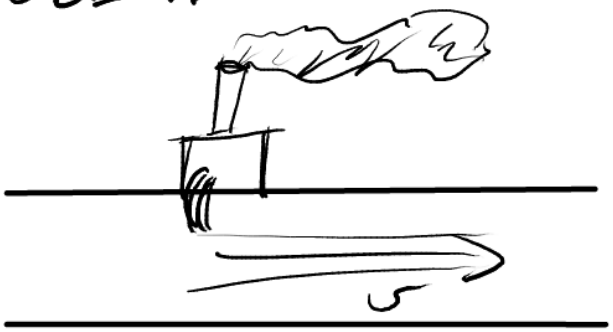
Moves towards the correct solution

Now $u = u(x(\xi,t),t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial t}$$

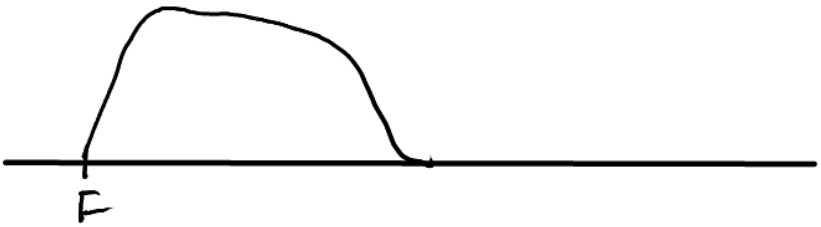
$$\frac{du}{dt} - \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} = f(u) \qquad \frac{\partial x}{\partial t} = \frac{1}{c}(M x_\xi)_\xi$$

Discretize this in space. Solve the remaining ODE's.



Pollution concentration $\rho_t = \varepsilon_\rho \rho_{xx} - v\rho_x + f_\rho(\rho, g) + g$

Oxygen concentration $g_t = \varepsilon_g g_{xx} - v g_x + f_g(\rho, g)$



In MATLAB: Example Burgers equation

DOC BURGERSODE

Huang, Ren, Russel 1994

Grid generation in 2D

Domain $\Omega \subseteq \mathbb{R}^d$, polygon domain

Triangulation: $\mathcal{T}_h = \{K \subseteq \mathbb{R}^d, \text{closed, polygons}\}$ satisfying

a) $\bigcup_{K \in \hat{\tau}_h} K = \Omega$

$\mathring{A}$ - interior of A

b) $\mathring{K} \neq \emptyset \quad K \in \tau_h$

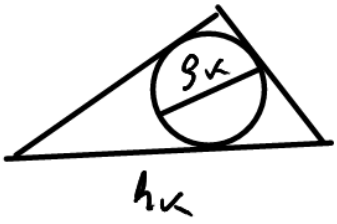
c) $\mathring{K} \cap \mathring{L} = \emptyset \quad \forall K, L \in \hat{\tau}_h$.

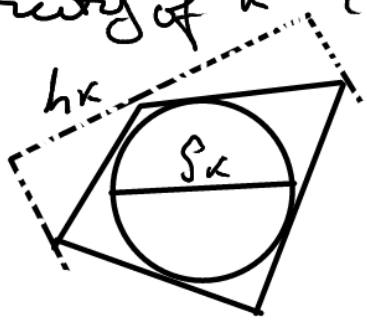
d) If $F = K \cap L$, then F is either $\emptyset$, a complete edge or a vertex $(K \neq L \in \hat{\tau}_h)$ (Conforming grid)

e) $\frac{h_K}{\rho_K} < \delta$ $\quad (\exists \delta : 5)$

h_K = diameter of K

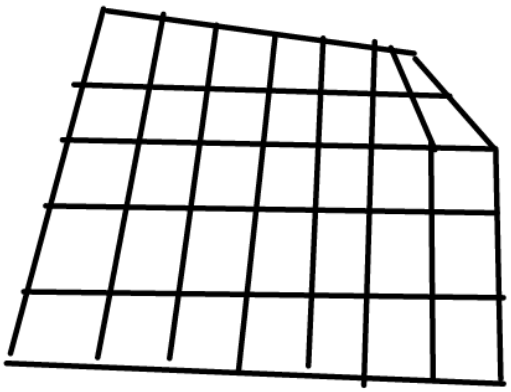
ρ_K = sphericity of K (Regular grid)





Structured grids

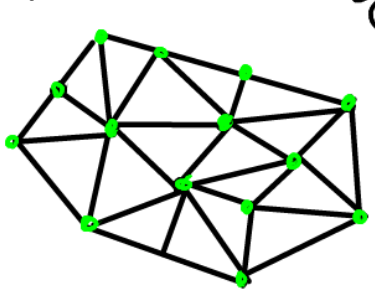
- Roughly speaking "Nice" mapping from a uniform grid on a square to the grid on Ω
- Usually quadrilateral elements.



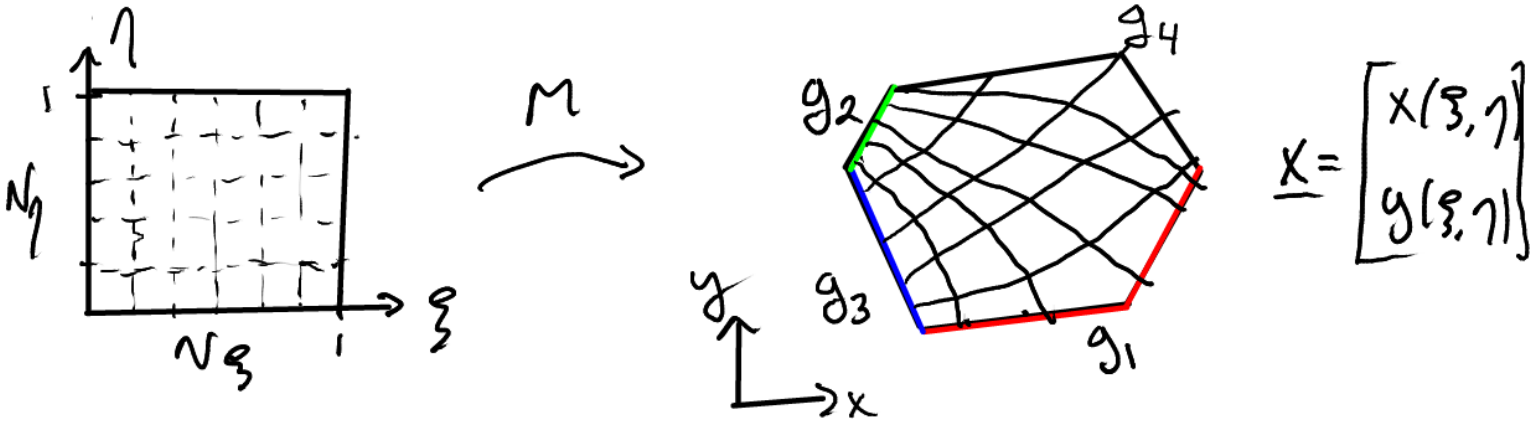
Advantages:
- Easy to use
- Easy to generate

Disadvantages:
- Less flexible.

Unstructured grid



More complicated to use, but more flexible.



$$\underline{X} = \begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{bmatrix}$$

Define $\underline{X}(\xi,0) = g_1(\xi)$ $\underline{X}(\xi,1) = g_2(\xi)$ $X(0,\eta) = g_3(\eta)$

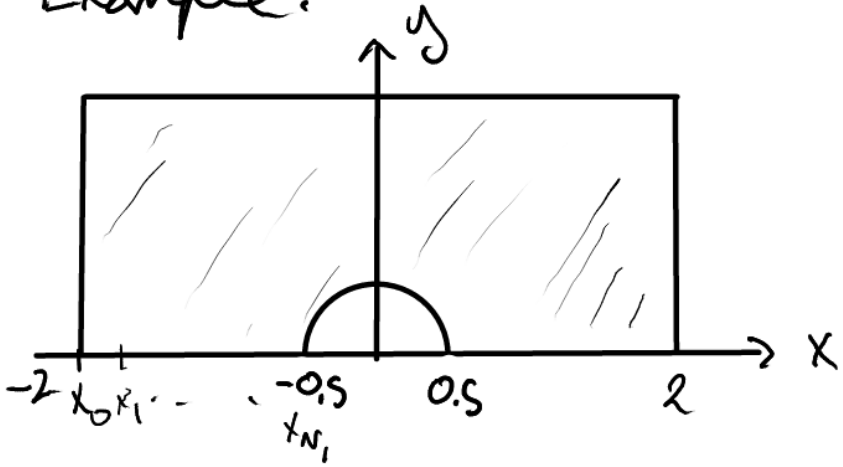
$\underline{X}(1,\eta) = g_4(\eta)$

Solve $-\Delta \underline{X} = 0$ or 2 independent equations

Solve it by some FD-method.

Works nice on a convex domain.

Example:



g_2, g_3, g_4 obvious.

Boundary g_1: $s = 3 + 0.5\pi$ (length)

Choose N_x, N_y.

$N_1 = \left\lceil \frac{1.5}{s} N_x \right\rceil$ $x_i = -2 + i \frac{1.5}{N_1}$ $0, \dots, N_1$. $y_i = 0$

$$\begin{cases} x_i = x_{N_1+j} = -0.5 \cos\left(j \cdot \frac{\pi}{N_2}\right), N_2 = N_x - 2N_1 \\ y_i = \sin\left(j \frac{\pi}{N_2}\right) \cdot 0.5 \end{cases}$$

Similarly for the last part

This does not work too well without making some changes.