1.1 Antiderivert

Oppgave 1.10

a) $a(t) = -0,48t + 2,4 \quad \Rightarrow \quad v(t) = -0,24t^2 + 2,4t + C$

$v(0) = 16 \Rightarrow -0,24 \cdot 0^2 + 2,4 \cdot 0 + C = 16 \Leftrightarrow C = 16 \qquad \underline{\underline{v(t) = -0,24t^2 + 2,4t + 16}}$

b) $v(5) = -0,24 \cdot 5^2 + 2,4 \cdot 5 + 16 = 22 \qquad$ <u>Farten etter 5 s er 22 m/s.</u>

c) $v(t) = -0,24t^2 + 2,4t + 16 \quad \Rightarrow \quad s(t) = -0,08t^3 + 1,2t^2 + 16t + C$

$s(0) = 0 \Rightarrow -0,08 \cdot 0^3 + 1,2 \cdot 0^2 + 16 \cdot 0 + C \Leftrightarrow C = 0 \qquad \underline{\underline{s(t) = -0,08t^3 + 1,2t^2 + 16t}}$

d) $s(5) = -0,08 \cdot 5^3 + 1,2 \cdot 5^2 + 16 \cdot 5 = 100 \qquad$ <u>Etter 5 s har bilen kommet 100 m.</u>

Oppgave 1.11

a) $\underline{\underline{F(x) = 2x^2 - 2x + C}}$ fordi $F'(x) = 4x - 2 = f(x)$

b) $\underline{\underline{F(x) = \frac{1}{3}x^3 + 2x^2 + C}}$ fordi $F'(x) = x^2 + 4x = f(x)$

c) $\underline{\underline{F(x) = \frac{1}{2}x^4 - \frac{1}{2}x^2 + C}}$ fordi $F'(x) = 2x^3 - x = f(x)$

d) $\underline{\underline{F(x) = x^4 - 3x^2 + x + C}}$ fordi $F'(x) = 4x^3 - 6x + 1 = f(x)$

Oppgave 1.12

$f(x) = 8x^3 + 12x^2 + 10x + 3 \quad \Rightarrow \quad F(x) = 2x^4 + 4x^3 + 5x^2 + 3x + C$

$F(-1) = 2 \Rightarrow 2 \cdot (-1)^4 + 4 \cdot (-1)^3 + 5 \cdot (-1)^2 + 3 \cdot (-1) + C = 2 \Leftrightarrow C = 2$

$\underline{\underline{F(x) = 2x^4 + 4x^3 + 5x^2 + 3x + 2}}$

Oppgave 1.13

a) $f(x) = e^x \quad \Rightarrow \quad \underline{\underline{F(x) = e^x + C}}$ fordi $F'(x) = e^x$

b) $f(x) = \frac{1}{x}, \ x > 0 \quad \Rightarrow \quad \underline{\underline{F(x) = \ln x + C}}$ fordi $F'(x) = \frac{1}{x}$

c) $f(x) = -\frac{1}{x^2} \quad \Rightarrow \quad \underline{\underline{F(x) = \frac{1}{x} + C}} \;= x^{-1} + C$ fordi $F'(x) = -1 \cdot x^{-2} = -\frac{1}{x^2}$

d) $f(x) = x + \frac{1}{x^2} \quad \Rightarrow \quad \underline{\underline{F(x) = \frac{1}{2}x^2 - \frac{1}{x} + C}} \;= \frac{1}{2}x^2 - x^{-1} + C$ fordi $F'(x) = x - (-1) \cdot x^{-2} = x + \frac{1}{x^2}$

1.2 Ubestemt integral

Oppgave 1.20

a) $\int(6x+5)\,dx = \underline{\underline{3x^2+5x+C}}$

b) $\int 4\,dx = \underline{\underline{4x+C}}$

c) $\int(3x^2+4x+1)\,dx = \underline{\underline{x^3+2x^2+x+C}}$

d) $\int(x^2-5x+6)\,dx = \underline{\underline{\frac{1}{3}x^3-\frac{5}{2}x^2+6x+C}}$

Oppgave 1.21

a) $\int\left(t-\frac{1}{t^2}\right)dt = \int\left(t-t^{-2}\right)dt = \frac{1}{2}t^2-\frac{1}{-1}t^{-1}+C = \underline{\underline{\frac{1}{2}t^2+\frac{1}{t}+C}}$

b) $\int\left(\frac{1}{x^2}-\frac{2}{x^3}\right)dx = \int\left(x^{-2}-2x^{-3}\right)dx = \frac{1}{-1}x^{-1}-2\cdot\frac{1}{-2}x^{-2}+C = \underline{\underline{-\frac{1}{x}+\frac{1}{x^2}+C}}$

c) $\int\left(2t-5\sqrt[3]{t^2}\right)dt = \int\left(2t-5t^{\frac{2}{3}}\right)dt = 2\cdot\frac{1}{2}t^2-5\cdot\frac{1}{\frac{5}{3}}t^{\frac{5}{3}}+C = \underline{\underline{t^2-3\cdot\sqrt[3]{t^5}+C}}$

d) $\int\frac{5}{2\sqrt{x}}\,dx = \int\frac{5}{2}x^{-\frac{1}{2}}dx = \frac{5}{2}\cdot\frac{1}{\frac{1}{2}}x^{\frac{1}{2}}+C = \underline{\underline{5\sqrt{x}+C}}$

Oppgave 1.22

$v(t) = 20+0{,}8t \quad\Rightarrow\quad s(t) = \int(20+0{,}8t)\,dt = 20t+0{,}4t^2+C$

$s(0)=0 \quad\Rightarrow\quad 20\cdot 0+0{,}4\cdot 0^2+C=0 \quad\Leftrightarrow\quad C=0 \quad\Rightarrow\quad s(t)=20t+0{,}4t^2$

$s(10) = 20\cdot 10+0{,}4\cdot 10^2 = 240$ <u>På de 10 sekundene kjører bilen 240 m.</u>

Oppgave 1.23

a) $f(x) = \int(4x-2)\,dx = 2x^2-2x+C$

$f(1)=2 \quad\Rightarrow\quad 2\cdot 1^2-2\cdot 1+C=2 \quad\Leftrightarrow\quad C=2 \quad\Rightarrow\quad \underline{\underline{f(x)=2x^2-2x+2}}$

b) $f(x) = \int \left(6x^2 + 5\right) dx = 6 \cdot \frac{1}{3} x^3 + 5x + C = 2x^3 + 5x + C$

$f(0) = 2 \quad \Rightarrow \quad 2 \cdot 0^3 + 5 \cdot 0 + C = 2 \quad \Leftrightarrow \quad C = 2 \quad \Rightarrow \quad \underline{\underline{f(x) = 2x^3 + 5x + 2}}$

c) $f'(x) = \int \left(12x - 6\right) dx = 6x^2 - 6x + C$

Bunnpunkt i $(1,0)$ $\Leftrightarrow$ $f'(1) = 0 \quad \Rightarrow \quad 6 \cdot 1^2 - 6 \cdot 1 + C = 0 \quad \Leftrightarrow \quad C = 0$

$\Rightarrow \quad f'(x) = 6x^2 - 6x$

$f(x) = \int \left(6x^2 - 6x\right) dx = 2x^3 - 3x^2 + C$

Punktet $(1,0)$ på grafen $\Rightarrow \quad f(1) = 0 \quad \Rightarrow \quad 2 \cdot 1^3 - 3 \cdot 1^2 + C = 0 \quad \Leftrightarrow \quad C = 1$

$\Rightarrow \quad \underline{\underline{f(x) = 2x^3 - 3x^2 + 1}}$

1.3 Integralet $\int \frac{1}{x}dx$

Oppgave 1.30

a) $\int \frac{3}{x}dx = \int 3\cdot\frac{1}{x}dx = \underline{\underline{3\ln|x| + C}}$

b) $\int \left(2x+1-\frac{2}{x}\right)dx = \underline{\underline{x^2 + x - 2\ln|x| + C}}$

c) $\int \left(6x^2 - 2x - 1 + \frac{1}{x} + \frac{1}{x^2}\right)dx = \int \left(6x^2 - 2x - 1 + \frac{1}{x} + x^{-2}\right)dx = 6\cdot\frac{1}{3}x^3 - 2\cdot\frac{1}{2}x^2 - x + \ln|x| + \frac{1}{-1}x^{-1} + C$

$= \underline{\underline{2x^3 - x^2 - x + \ln|x| - \frac{1}{x} + C}}$

Oppgave 1.31

a) $\int \frac{x+1}{x}dx = \int \left(\frac{x}{x} + \frac{1}{x}\right)dx = \int \left(1 + \frac{1}{x}\right) = \underline{\underline{x + \ln|x| + C}}$

b) $\int \frac{2x^2+x-2}{x}dx = \int \left(\frac{2x^2}{x} + \frac{x}{x} - \frac{2}{x}\right)dx = \int \left(2x+1-\frac{2}{x}\right) = \underline{\underline{x^2 + x - 2\ln|x| + C}}$

Oppgave 1.32

a) $\int \frac{1}{x+1}dx \overset{u(x)=x+1}{=} \ln|x+1|\cdot\frac{1}{1} = \underline{\underline{\ln|x+1| + C}}$

b) $\int \frac{1}{2x+1}dx \overset{u(x)=2x+1}{=} \ln|2x+1|\cdot\frac{1}{2} = \underline{\underline{\frac{1}{2}\ln|2x+1| + C}}$

Oppgave 1.33

a) $(x\ln x - x)' = \left(1\cdot\ln x + x\cdot\frac{1}{x}\right) - 1 = \ln x + 1 - 1 = \underline{\underline{\ln x}}$

b) $\int \ln x dx = \underline{\underline{x\ln x - x + C}}$ fordi $(x\ln x - x + C)' = \ln x$.

1.4 Integrasjon av eksponentialfunksjoner

Oppgave 1.40

a) $\int 2e^x dx = \underline{\underline{2e^x + C}}$

b) $\int (6x - 3e^x)\, dx = \underline{\underline{3x^2 - 3e^x + C}}$

c) $\int (4e^{2x} + 9e^{3x})\, dx = 4 \cdot \frac{1}{2} e^{2x} + 9 \cdot \frac{1}{3} e^{3x} + C = \underline{\underline{2e^{2x} + 3e^{3x} + C}}$

d) $\int (e^x + e^{-x})\, dx = e^x + \frac{1}{-1} e^{-x} + C = \underline{\underline{e^x - e^{-x} + C}}$

Oppgave 1.41

a) $\int (3 \cdot 2^x)\, dx = 3 \cdot \frac{1}{\ln 2} 2^x + C = \underline{\underline{\frac{3}{\ln 2} 2^x + C}}$

b) $\int (2 - 3^x)\, dx = \underline{\underline{2x - \frac{1}{\ln 3} 3^x + C}}$

c) $\int (\ln 2 \cdot 2^x + \ln 3 \cdot 3^x)\, dx = \ln 2 \cdot \frac{1}{\ln 2} 2^x + \ln 3 \cdot \frac{1}{\ln 3} 3^x + C = \underline{\underline{2^x + 3^x + C}}$

Oppgave 1.42

a) $\int (5400 \cdot e^{0{,}08x})\, dx = 5400 \cdot \frac{1}{0{,}08} e^{0{,}08x} + C = \underline{\underline{67\,500 \cdot e^{0{,}08x} + C}}$

b) $\int (12800 \cdot 1{,}05^x)\, dx = 12800 \cdot \frac{1}{\ln 1{,}05} \cdot 1{,}05^x + C = \underline{\underline{262\ 348 \cdot 1{,}05^x + C}}$

Oppgave 1.43

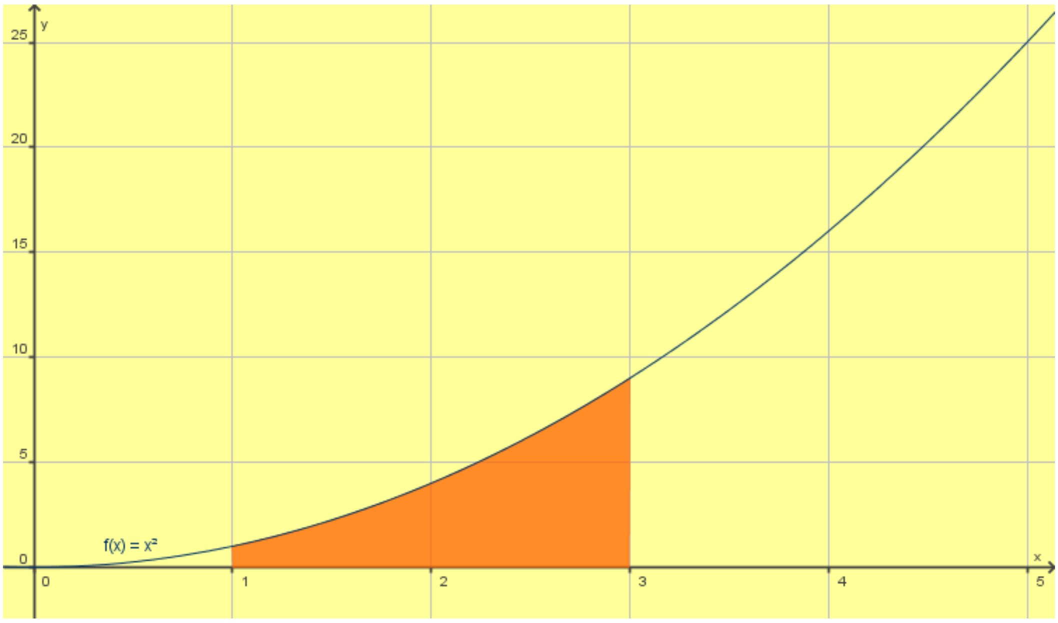
a) $\left(e^{x^2}\right)' = e^{x^2} \cdot 2x = \underline{\underline{2xe^{x^2}}}$

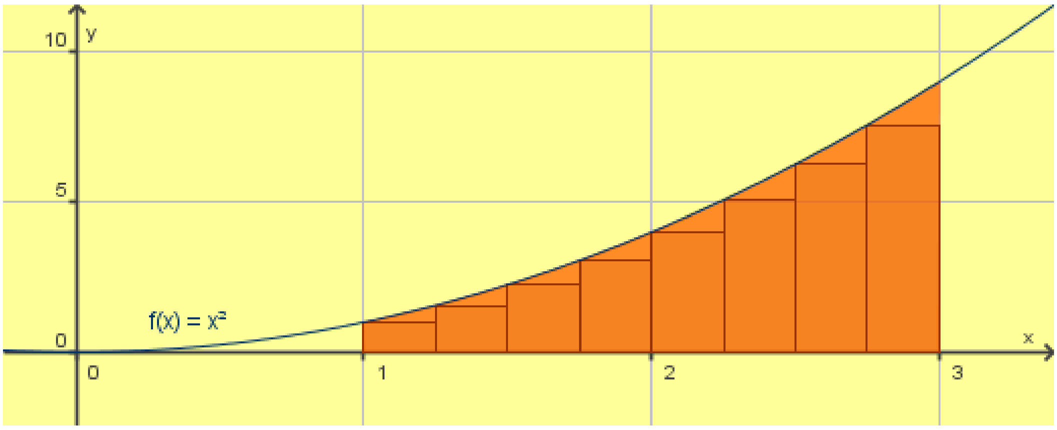
b) $\int xe^{x^2} dx = \underline{\underline{\frac{1}{2} e^{x^2} + C}}$ fordi $\left(\frac{1}{2} e^{x^2} + C\right)' = \underline{\underline{xe^{x^2}}}$

1.5 Bestemt integral som grense for en sum

Oppgave 1.50

a)





b)

$$\Delta x = \frac{3-1}{8} = \frac{1}{4}$$

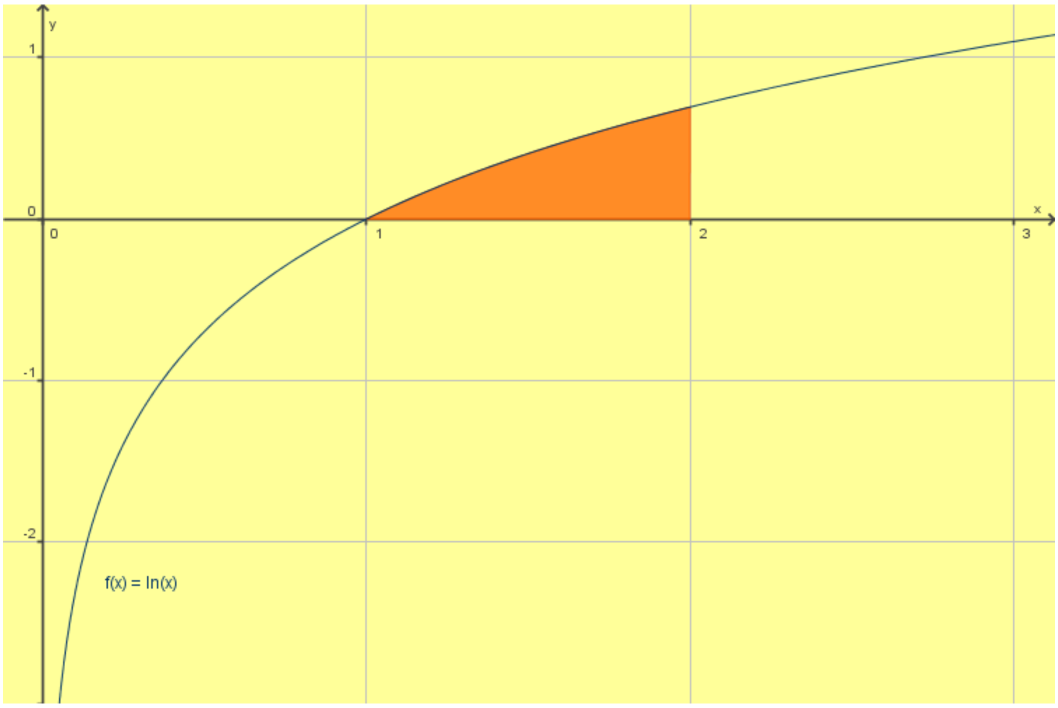
$$S_8 = f(1)\cdot\tfrac{1}{4} + f\left(\tfrac{5}{4}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{6}{4}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{7}{4}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{8}{4}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{9}{4}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{10}{4}\right)\cdot\tfrac{1}{4} + f\left(\tfrac{11}{4}\right)\cdot\tfrac{1}{4}$$

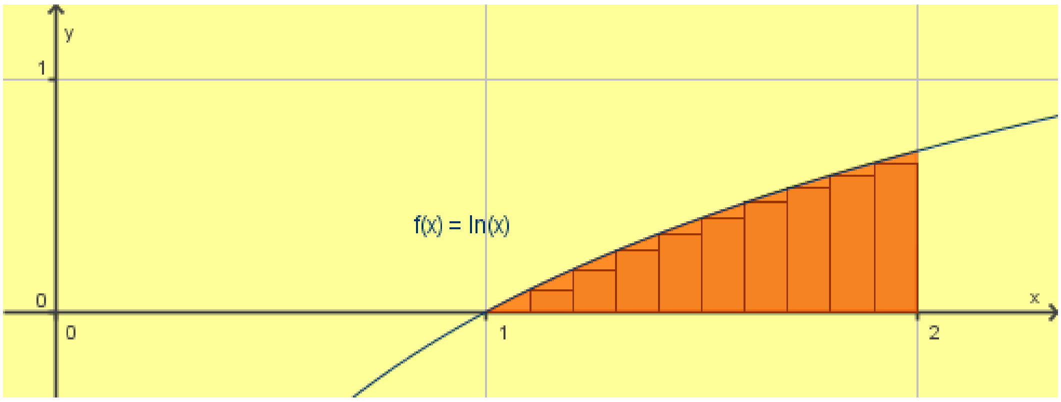
$$= 1^2\cdot\tfrac{1}{4} + \left(\tfrac{5}{4}\right)^2\cdot\tfrac{1}{4} + \left(\tfrac{6}{4}\right)^2\cdot\tfrac{1}{4} + \left(\tfrac{7}{4}\right)^2\cdot\tfrac{1}{4} + \left(\tfrac{8}{4}\right)^2\cdot\tfrac{1}{4} + \left(\tfrac{9}{4}\right)^2\cdot\tfrac{1}{4} + \left(\tfrac{10}{4}\right)^2\cdot\tfrac{1}{4} + \left(\tfrac{11}{4}\right)^2\cdot\tfrac{1}{4}$$

$$= \tfrac{1}{4} + \tfrac{25}{64} + \tfrac{36}{64} + \tfrac{49}{64} + \tfrac{64}{64} + \tfrac{81}{64} + \tfrac{100}{64} + \tfrac{121}{64} = \underline{\underline{\tfrac{492}{64}}} \approx \underline{\underline{7,69}}$$

Oppgave 1.51

a)
+
b)





c)

$$\Delta x = \frac{2-1}{10} = \frac{1}{10}$$

$$S_8 = f\left(1\right)\cdot\tfrac{1}{10} + f\left(\tfrac{11}{10}\right)\cdot\tfrac{1}{10} + f\left(\tfrac{12}{10}\right)\cdot\tfrac{1}{10} + \ldots + f\left(\tfrac{17}{10}\right)\cdot\tfrac{1}{10} + f\left(\tfrac{18}{10}\right)\cdot\tfrac{1}{10} + f\left(\tfrac{19}{10}\right)\cdot\tfrac{1}{10}$$

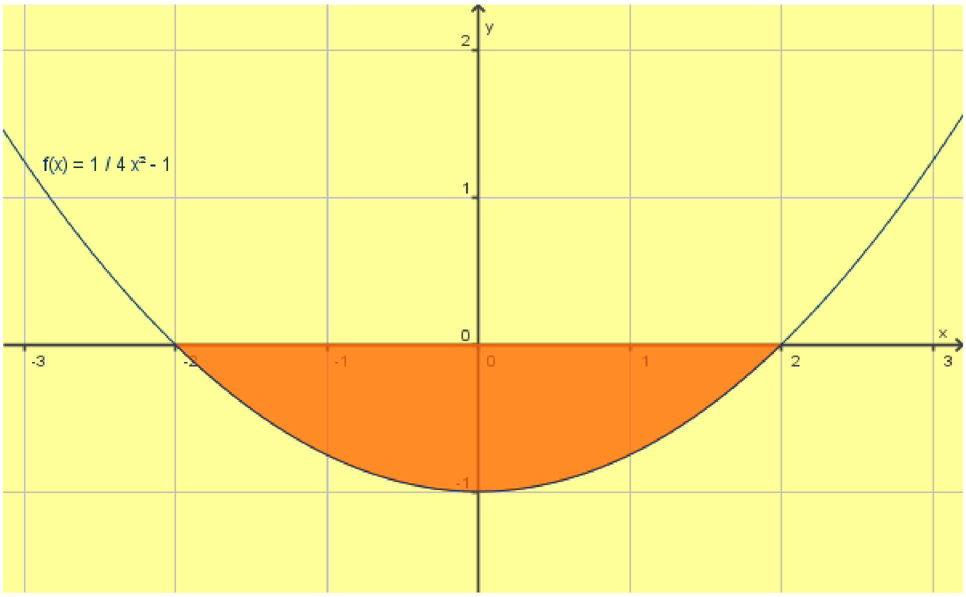
$$= \ln 1\cdot\tfrac{1}{10} + \ln\left(\tfrac{11}{10}\right)\cdot\tfrac{1}{10} + \ln\left(\tfrac{12}{10}\right)\cdot\tfrac{1}{10} + .. + \ln\left(\tfrac{17}{10}\right)\cdot\tfrac{1}{10} + \ln\left(\tfrac{18}{10}\right)\cdot\tfrac{1}{10} + \ln\left(\tfrac{19}{10}\right)\cdot\tfrac{1}{10} \approx \underline{\underline{0,35}}$$

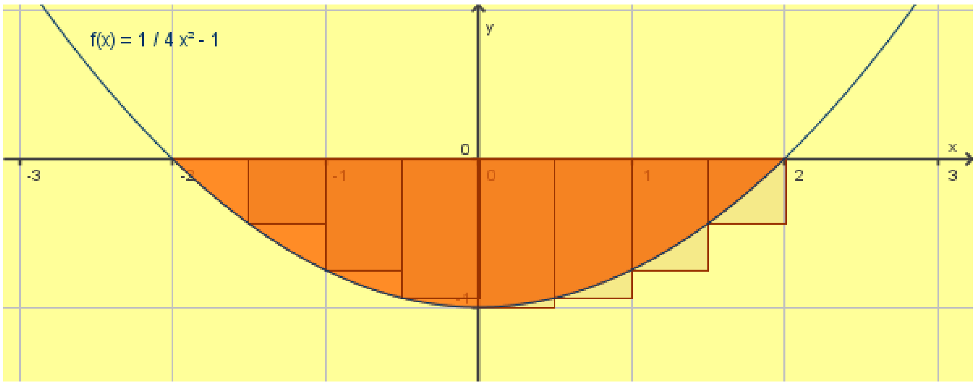
d)

$$\int_1^2 \ln x dx \approx \underline{\underline{0,35}}$$

Oppgave 1.52

a)
+
b)





c)

$$\Delta x = \frac{2-(-2)}{8} = \frac{1}{2}$$

$$S_8 = f(-2)\cdot\tfrac{1}{2} + f\left(-\tfrac{3}{2}\right)\cdot\tfrac{1}{2} + f\left(-\tfrac{2}{2}\right)\cdot\tfrac{1}{2} + f\left(-\tfrac{1}{2}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{0}{2}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{1}{2}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{2}{2}\right)\cdot\tfrac{1}{2} + f\left(\tfrac{3}{2}\right)\cdot\tfrac{1}{2}$$

$$= \left(\tfrac{1}{4}\cdot(-2)^2 - 1\right)\cdot\frac{1}{2} + \left(\tfrac{1}{4}\cdot\left(-\tfrac{3}{2}\right)^2 - 1\right)\cdot\frac{1}{2} + .. + \left(\frac{1}{4}\cdot\left(\tfrac{1}{2}\right)^2 - 1\right)\cdot\frac{1}{2} + \left(\tfrac{1}{4}\cdot 1^2 - 1\right)\cdot\frac{1}{2} + \left(\frac{1}{4}\cdot\left(\tfrac{3}{2}\right)^2 - 1\right)\cdot\frac{1}{2}$$

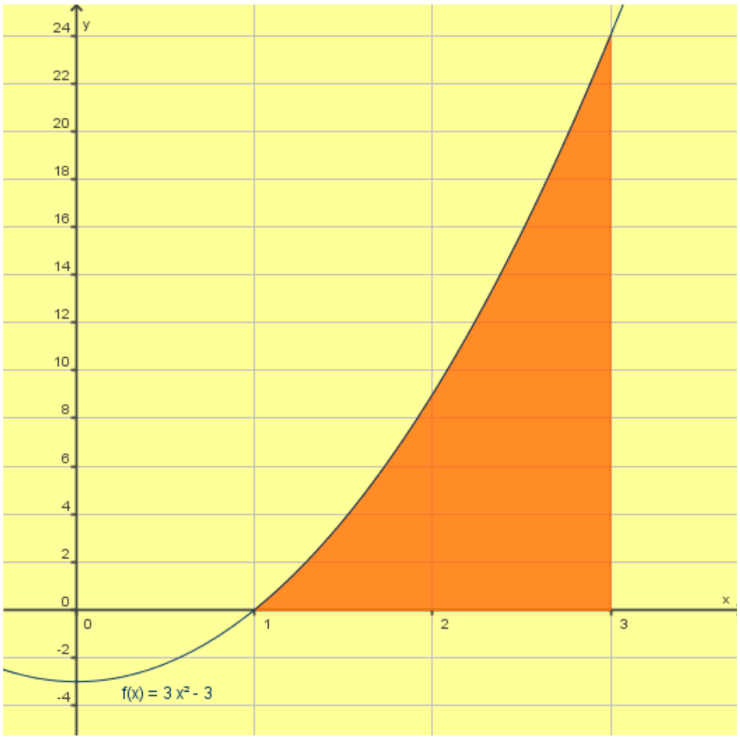
$$\approx -2{,}63$$

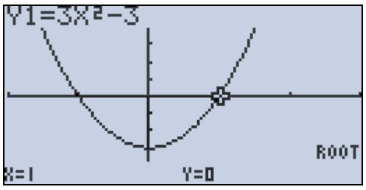
$$\underline{\underline{\int_{-2}^{2}\left(\tfrac{1}{4}x^2 - 1\right)dx \approx -2{,}63}}$$

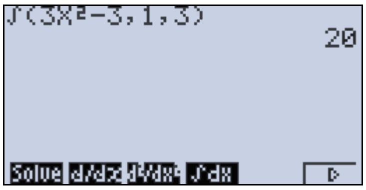
d)

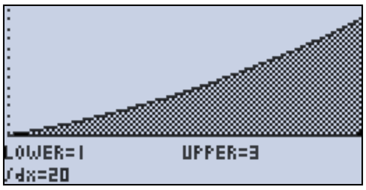
$$F = -\int_{-2}^{2}\left(\tfrac{1}{4}x^2 - 1\right)dx \approx \underline{\underline{2{,}63}}$$

Oppgave 1.53



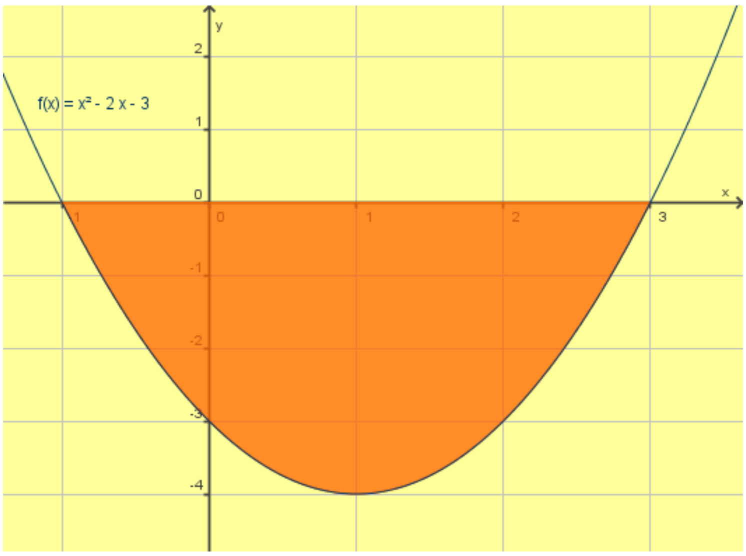


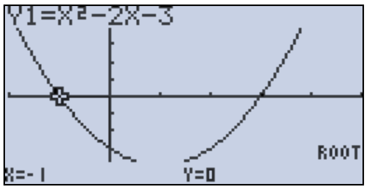


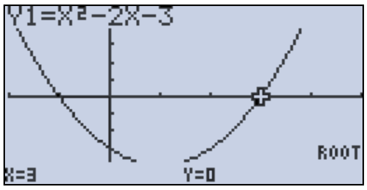


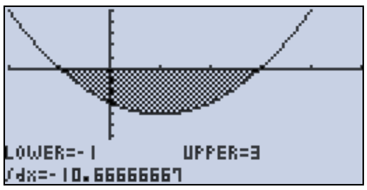
Avgrenset areal er 20

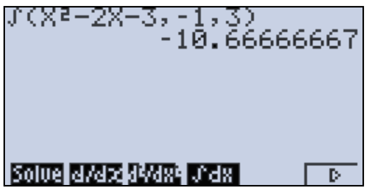
Oppgave 1.54









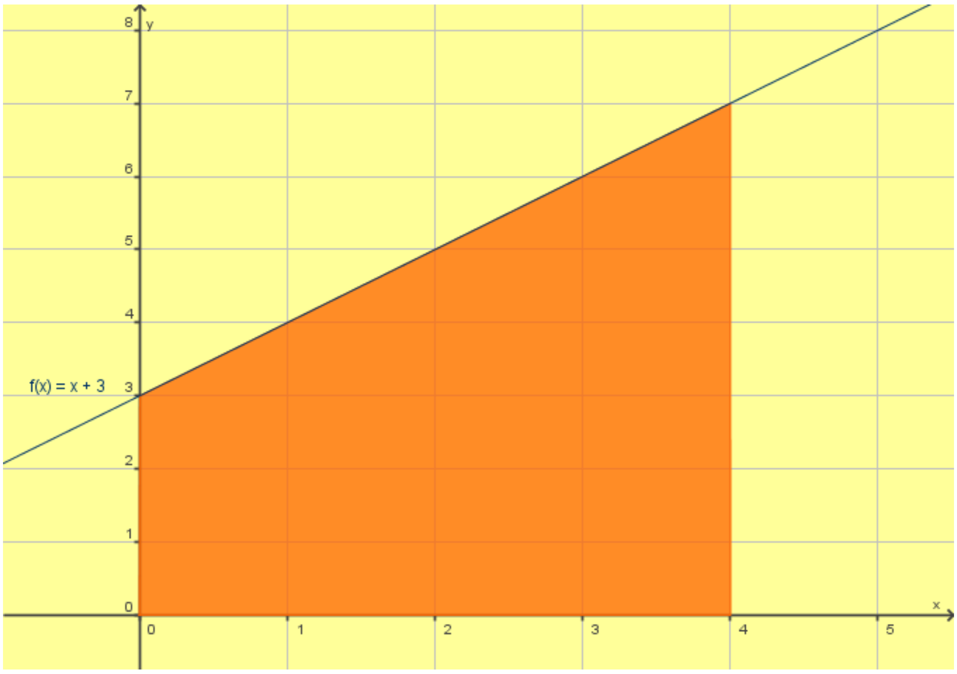


Avgrenset areal er $\frac{32}{3} \approx 10{,}67$

1.6 Bestemt integral og antiderivasjon

Oppgave 1.60

a)



b) $A = \dfrac{f(0)+f(4)}{2}\cdot \Delta x = \dfrac{3+7}{2}\cdot 4 = \underline{\underline{20}}$

c) $A = \int_0^4 (x+3)\,dx = \left[\tfrac{1}{2}x^2 + 3x\right]_0^4 = \left(\tfrac{1}{2}\cdot 4^2 + 3\cdot 4\right) - 0 = 8 + 12 = \underline{\underline{20}}$

Oppgave 1.61

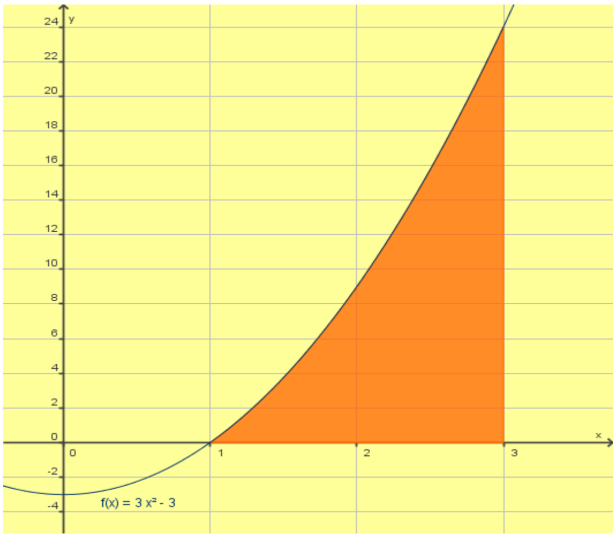
a) $\int_{-1}^{2} (3x^2 + 2x)\,dx = \left[x^3 + x^2\right]_{-1}^{2} = \left(2^3 + 2^2\right) - \left((-1)^3 + (-1)^2\right) = 8 + 4 + 1 - 1 = \underline{\underline{12}}$

b) $\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e^1 - e^0 = \underline{\underline{e-1}}$

c) $\int_0^5 2^x\,dx = \left[\tfrac{1}{\ln 2}\cdot 2^x\right]_0^5 = \tfrac{1}{\ln 2}\cdot 2^5 - \tfrac{1}{\ln 2}\cdot 2^0 = \tfrac{32-1}{\ln 2} = \underline{\underline{\tfrac{31}{\ln 2}}}$

d) $\int_1^e \tfrac{1}{x}\,dx = \left[\ln|x|\right]_1^e = \ln e - \ln 1 = 1 - 0 = \underline{\underline{1}}$

Oppgave 1.62



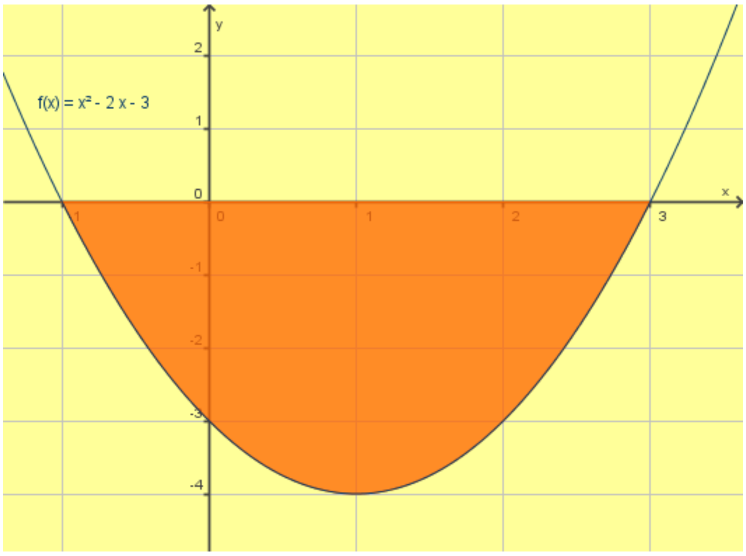
Nedre integrasjonsgrense bestemt av $f(x)=0$

$$\Rightarrow \quad 3x^2-3=0 \quad \Leftrightarrow \quad x^2=1 \quad \Leftrightarrow \quad x=\pm\sqrt{1}=\pm 1$$

$$A=\int_1^3\left(3x^2-3\right)dx=\left[x^3-3x\right]_1^3$$

$$=\left(3^3-3\cdot 3\right)-\left(1^3-3\cdot 1\right)=27-9-1+3=\underline{\underline{20}}$$

Oppgave 1.63



Integrasjonsgrensene bestemt av $f(x)=0$

$$\Rightarrow \quad x^2-2x-3=0 \quad \Leftrightarrow \quad x=\frac{-(-2)\pm\sqrt{(-2)^2-4\cdot 1\cdot(-3)}}{2\cdot 1}$$

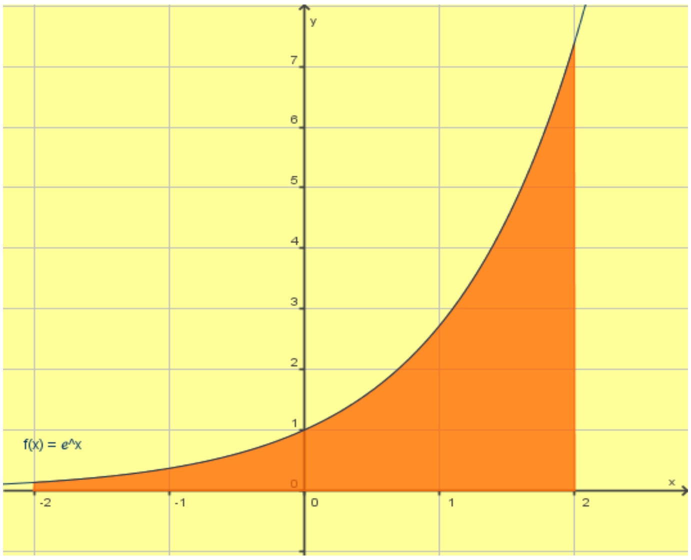
$$\Leftrightarrow \quad x_1=\frac{2-\sqrt{16}}{2}=\frac{2-4}{2}=-1 \quad \wedge \quad x_2=\frac{2+4}{2}=3$$

$$A=-\int_{-1}^3\left(x^2-2x-3\right)dx=\int_{-1}^3\left(-x^2+2x+3\right)dx=\left[-\tfrac{1}{3}x^3+x^2+3x\right]_{-1}^3$$

$$=\left(-\tfrac{1}{3}\cdot 3^3+3^2+3\cdot 3\right)-\left(-\tfrac{1}{3}\cdot(-1)^3+(-1)^2+3\cdot(-1)\right)=-9+9+9-\tfrac{1}{3}-1+3=\underline{\underline{\tfrac{32}{3}}}$$

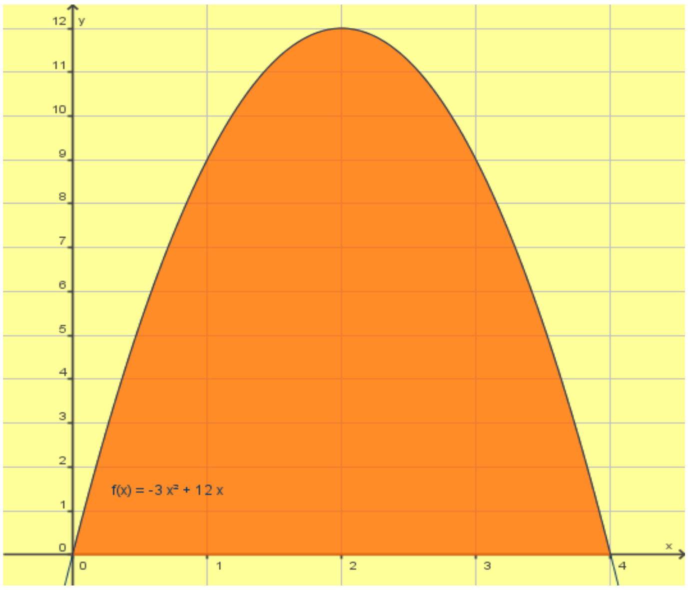
Oppgave 1.64

a)



$$A=\int_{-2}^{2} e^{x}\,dx=\left[e^{x}\right]_{-2}^{2}=\underline{\underline{e^{2}-e^{-2}}}=\underline{\underline{e^{2}-\frac{1}{e^{2}}}}$$

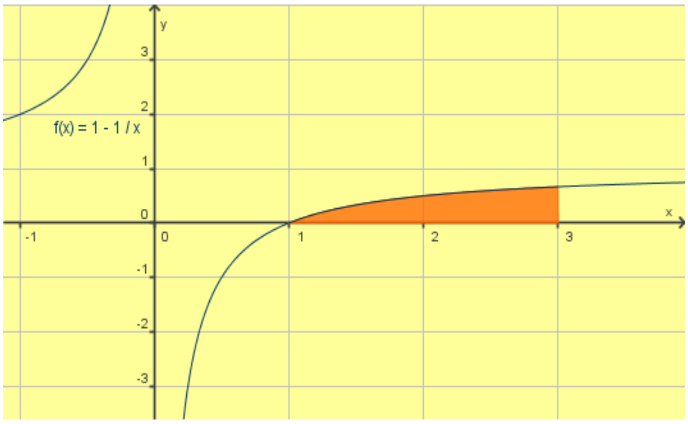
b)



$$A=\int_{0}^{4}\left(-3x^{2}+12x\right)dx=\left[-x^{3}+6x^{2}\right]_{0}^{4}$$

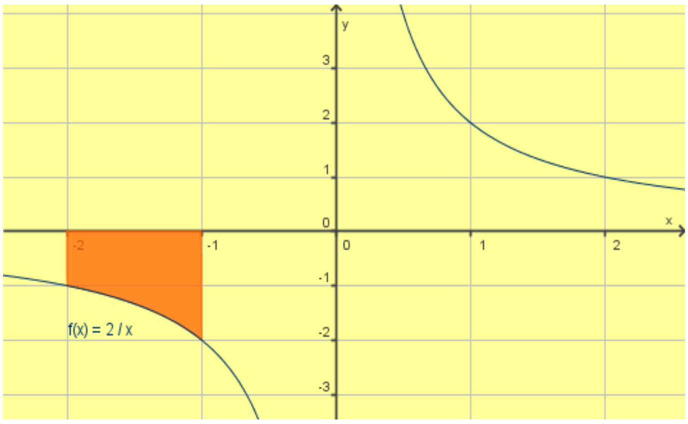
$$=\left(-4^{3}+6\cdot 4^{2}\right)-0=-64+96=\underline{\underline{32}}$$

c)



$$A=\int_1^3\left(1-\tfrac{1}{x}\right)dx=\left[x-\ln|x|\right]_1^3$$
$$=(3-\ln 3)-(1-\ln 1)$$
$$=3-\ln 3-1=\underline{\underline{2-\ln 3}}$$

d)

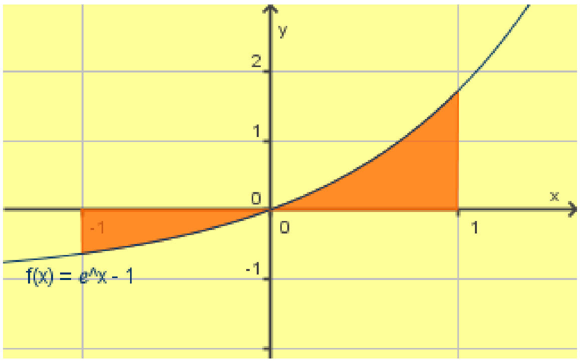


$$A=-\int_{-2}^{-1}\frac{2}{x}dx=\left[-2\ln|x|\right]_{-2}^{-1}$$
$$=-2\ln 1-(-2\ln 2)$$
$$=\underline{\underline{2\ln 2}}=\ln 2^2=\underline{\underline{\ln 4}}$$

1.7 Integrasjon og areal

Oppgave 1.70

a)



Nullpunktet gitt ved $f(x)=0$

$\Rightarrow \quad e^x-1=0 \quad \Leftrightarrow \quad e^x=1 \quad \Leftrightarrow \quad x=\ln 1=0$

$$A_{under}=-\int_{-1}^{0}\left(e^x-1\right)dx=\int_{-1}^{0}\left(1-e^x\right)dx=\left[x-e^x\right]_{-1}^{0}$$

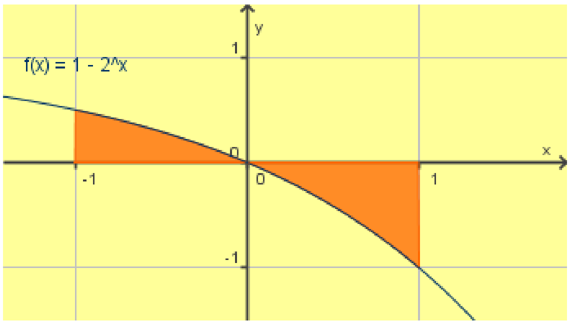
$$=\left(0-e^0\right)-\left(-1-e^{-1}\right)=-1+1+\tfrac{1}{e}=\tfrac{1}{e}$$

$$A_{over}=\int_{0}^{1}\left(e^x-1\right)dx=\left[e^x-x\right]_{0}^{1}$$

$$=\left(e^1-1\right)-\left(e^0-0\right)=e-1-1=e-2$$

$$\underline{\underline{A_{totalt}=\frac{1}{e}+e-2}}\approx\underline{\underline{1,09}}$$

b)



Nullpunktet gitt ved $f(x)=0$

$\Rightarrow \quad 1-2^x=0 \quad \Leftrightarrow \quad 2^x=1 \quad \Leftrightarrow \quad x=\frac{\ln 1}{\ln 2}=0$

$$A_{over}=\int_{-1}^{0}\left(1-2^x\right)dx=\left[x-\tfrac{1}{\ln 2}\cdot 2^x\right]_{-1}^{0}$$

$$=\left(0-\tfrac{2^0}{\ln 2}\right)-\left(-1-\tfrac{2^{-1}}{\ln 2}\right)=-\tfrac{1}{\ln 2}+1+\tfrac{1}{2\ln 2}$$

$$=1-\tfrac{1}{2\ln 2}$$

$$A_{under}=-\int_{0}^{1}\left(1-2^x\right)dx=\int_{0}^{1}\left(2^x-1\right)dx=\left[\tfrac{1}{\ln 2}\cdot 2^x-x\right]_{0}^{1}$$

$$=\left(\tfrac{2^1}{\ln 2}-1\right)-\left(\tfrac{2^0}{\ln 2}-0\right)=\tfrac{2}{\ln 2}-1-\tfrac{1}{\ln 2}=\tfrac{1}{\ln 2}-1$$

$$A_{over}+A_{under}=\left(1-\tfrac{1}{2\ln 2}\right)+\left(\tfrac{1}{\ln 2}-1\right)$$

$$=1-\tfrac{1}{2\ln 2}+\tfrac{1}{\ln 2}-1=\underline{\underline{\tfrac{1}{2\ln 2}}}\approx\underline{\underline{0,72}}$$

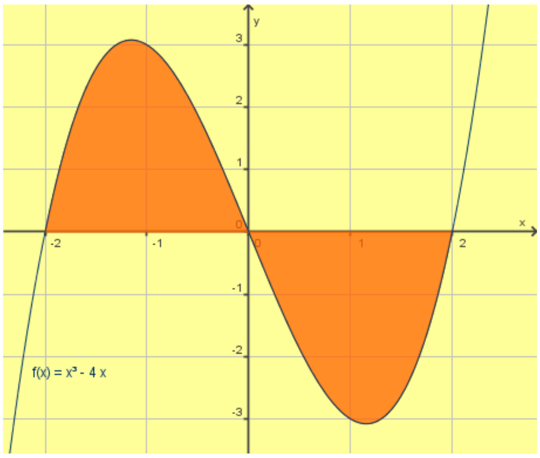
Oppgave 1.71

a) Nullpunkter gitt ved $f(x)=0$

$\Rightarrow\ x^3-4x=0 \ \Leftrightarrow\ x\cdot\left(x^2-4\right)=0 \ \Leftrightarrow\ x=0 \ \vee\ x^2-4=0 \ \Leftrightarrow\ x=0 \ \vee\ x=\pm\sqrt{4}=\pm 2$

Nullpunktene til f er $\underline{\underline{x=-2\ ,\ x=0\ \text{og}\ x=2}}$

b)



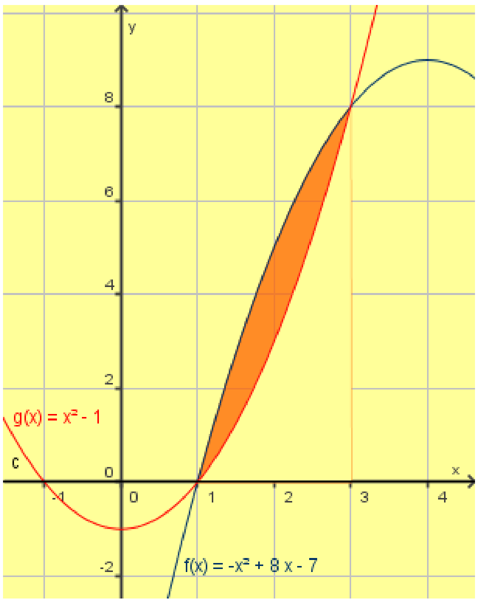
c)

$$A_{over}=\int_{-2}^{0}\left(x^3-4x\right)dx=\left[\tfrac{1}{4}x^4-2x^2\right]_{-2}^{0}$$
$$=0-\left(\tfrac{1}{4}\cdot(-2)^4-2\cdot(-2)^2\right)=-4+8=4$$

$$A_{under}=-\int_{0}^{2}\left(x^3-4x\right)dx=\int_{0}^{2}\left(-x^3+4x\right)dx$$
$$=\left[-\tfrac{1}{4}x^4+2x^2\right]_{0}^{2}=\left(-\tfrac{1}{4}\cdot 2^4+2\cdot 2^2\right)-0$$
$$=-4+8=4$$

$$A_{over}+A_{under}=4+4=\underline{\underline{8}}$$

Oppgave 1.72



Integrasjonsgrenser: $f(x)=g(x) \ \Leftrightarrow\ f(x)-g(x)=0$

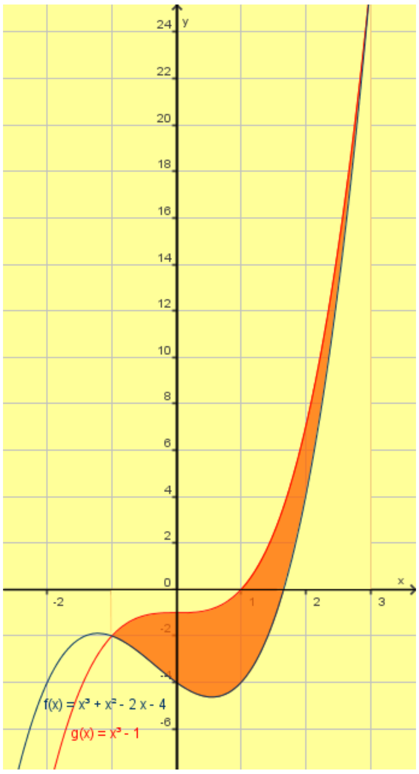
$$\left(-x^2+8x-7\right)-\left(x^2-1\right)=0 \Leftrightarrow \underbrace{-2x^2+8x-6}_{f(x)-g(x)}=0 \ \Leftrightarrow$$

$$x=\frac{-8\pm\sqrt{8^2-4\cdot(-2)\cdot(-6)}}{2\cdot(-2)}=\frac{-8\pm\sqrt{16}}{-4} \ \Leftrightarrow$$

$$x=\frac{-8+4}{-4}=1 \ \vee\ x=\frac{-8-4}{-4}=3$$

$$A=\int_{1}^{3}\left(f(x)-g(x)\right)dx=\int_{1}^{3}\left(-2x^2+8x-6\right)dx=$$
$$=\left[-\tfrac{2}{3}x^3+4x^2-6x\right]_{1}^{3}$$
$$=\left(-\tfrac{2}{3}\cdot 3^3+4\cdot 3^2-6\cdot 3\right)-\left(-\tfrac{2}{3}\cdot 1^3+4\cdot 1^2-6\cdot 1\right)$$
$$=-18+36-18+\tfrac{2}{3}-4+6=\underline{\underline{\tfrac{8}{3}}}$$

Oppgave 1.73



Integrasjonsgrenser: $g(x) = f(x) \Leftrightarrow g(x) - f(x) = 0$

$$\left(x^3 - 1\right) - \left(x^3 + x^2 - 2x - 4\right) = 0 \Leftrightarrow \underbrace{-x^2 + 2x + 3}_{g(x)-f(x)} = 0 \Leftrightarrow$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4\cdot(-1)\cdot 3}}{2\cdot(-1)} = \frac{-2 \pm \sqrt{16}}{-2} \Leftrightarrow$$

$$x = \frac{-2+4}{-2} = -1 \quad \vee \quad x = \frac{-2-4}{-2} = 3$$

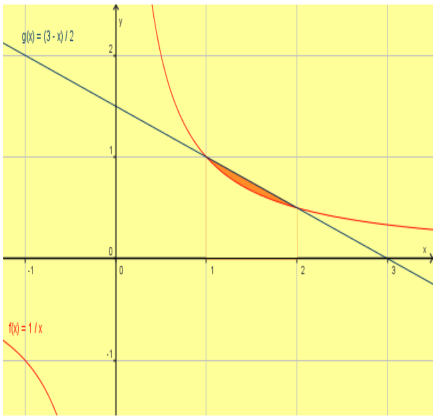
$$A = \int_{-1}^{3} \left(g(x) - f(x)\right) dx = \int_{-1}^{3} \left(-x^2 + 2x + 3\right) dx =$$

$$= \left[-\tfrac{1}{3}x^3 + x^2 + 3x\right]_{-1}^{3}$$

$$= \left(-\tfrac{1}{3}\cdot 3^3 + 3^2 + 3\cdot 3\right) - \left(-\tfrac{1}{3}\cdot(-1)^3 + (-1)^2 + 3\cdot(-1)\right)$$

$$= -9 + 9 + 9 - \tfrac{1}{3} - 1 + 3 = \underline{\underline{\frac{32}{3}}}$$

Oppgave 1.74



Integrasjonsgrenser: $g(x) = f(x) \iff g(x) - f(x) = 0$

$$\frac{3-x}{2} - \frac{1}{x} = 0 \iff 3x - x^2 - 2 = 0 \iff$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4\cdot(-1)\cdot(-2)}}{2\cdot(-1)} = \frac{-3 \pm \sqrt{1}}{-2} \iff$$

$$x = \frac{-3+1}{-2} = 1 \quad \vee \quad x = \frac{-3-1}{-2} = 2$$

$$A = \int_1^2 \left(g(x) - f(x)\right)dx = \int_1^2 \left(\frac{3-x}{2} - \frac{1}{x}\right)dx = \int_1^2 \left(\frac{3}{2} - \frac{1}{2}x - \frac{1}{x}\right)dx$$

$$= \left[\frac{3}{2}x - \frac{1}{4}x^2 - \ln|x|\right]_1^2$$

$$= \left(\frac{3}{2}\cdot 2 - \frac{1}{4}\cdot 2^2 - \ln 2\right) - \left(\frac{3}{2}\cdot 1 - \frac{1}{4}\cdot 1^2 - \ln 1\right)$$

$$= 3 - 1 - \ln 2 - \frac{3}{2} + \frac{1}{4} = \underline{\underline{\frac{3}{4} - \ln 2}}$$

1.8 Integral og samlet resultat

Oppgave 1.80

a) $f(52) = 250 \cdot e^{0{,}01 \cdot 52} \approx 421$ Ukesproduksjonen om et år vil være 421 enheter.

b) $\int_0^{52} 250 \cdot e^{0{,}01t} dt = \left[\frac{250}{0{,}01} e^{0{,}01t}\right]_0^{52} = 25000 \cdot e^{0{,}52} - 25000 \cdot e^0 \approx 17\,051$

Årsproduksjonen det første året blir 17 051 enheter.

c) Gjennomsnittlig ukeproduksjon det første året: $\frac{17051 \text{ enheter}}{52} \approx \underline{\underline{328 \text{ enheter}}}$

Oppgave 1.81

a) Årlig økning på 1,2% $\Rightarrow$ vekstfaktor 1,012

Fødselstall om 30 år: $50\,000 \cdot 1{,}012^{30} \approx \underline{\underline{71500}}$

b) $\int_0^{30} 50\,000 \cdot 1{,}012^x dx = \left[\frac{50000}{\ln 1{,}012} \cdot 1{,}012^x\right]_0^{30} = \frac{50000}{\ln 1{,}012} \cdot 1{,}012^{30} - \frac{50000}{\ln 1{,}012} \cdot 1{,}012^0 \approx 1803500$

Totalt blir det på de 30 årene født ca. 1,8 millioner barn.

c) Gjennomsnittlig årlig fødselstall: $\frac{1803500}{30} = \underline{\underline{60100}}$

Oppgave 1.82

Totale matutgifter: $\int_0^{20} \left(-150x^2 + 4000x + 70000\right) dx = \left[-50x^3 + 2000x^2 + 70000x\right]_0^{20}$

$= \left(-50 \cdot 20^3 + 2000 \cdot 20^2 + 70000 \cdot 20\right) - 0 = 1800000$

Gjennomsnittlige matutgifter per år: $\frac{1800000 \text{kr}}{20} = \underline{\underline{90\,000 \text{kr}}}$

Oppgave 1.83

Martes lønnsfunksjon: $M(t) = 240\,000 + 10\,000t$

Sondres lønnsfunksjon: $S(t) = 200\,000 \cdot 1,05^t$

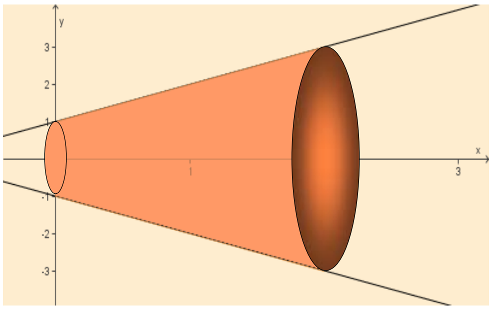
Martes totalinntekt: $\int_0^{25} (240\,000 + 10\,000t)dt = \left[240000t + 5000t^2\right]_0^{25} = 9125\,000$

Sondres totalinntekt: $\int_0^{25} 200\,000 \cdot 1,05^t\, dt = \left[\frac{200000 \cdot 1,05^t}{\ln 1,05}\right]_0^{25} = \frac{200000}{\ln 1,05}\left(1,05^{25} - 1,05^0\right) \approx 9\,782\,000$

Sondre vil tjene mest.

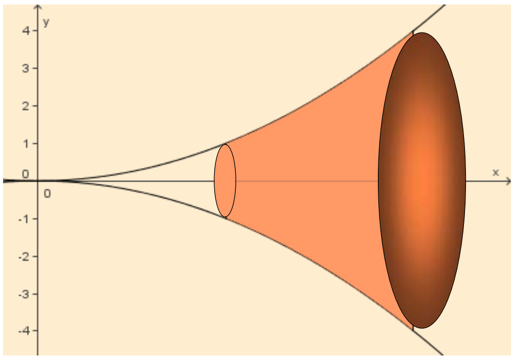
1.9 Integrasjon og volum

Oppgave 1.90



$$V=\pi\int_0^2 (x+1)^2\,dx=\pi\int_0^2\left(x^2+2x+1\right)dx$$
$$=\pi\cdot\left[\tfrac{1}{3}x^3+x^2+x\right]_0^2$$
$$=\pi\cdot\left(\tfrac{1}{3}\cdot 2^3+2^2+2\right)=\underline{\underline{\frac{26\pi}{3}}}$$

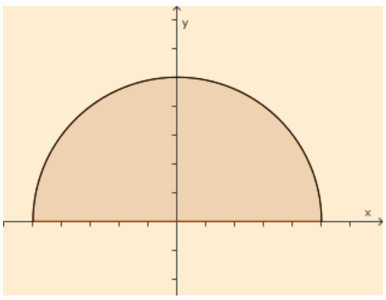
Oppgave 1.91



$$V=\pi\int_1^2\left(x^2\right)^2dx=\pi\int_1^2 x^4dx=\pi\cdot\left[\tfrac{1}{5}x^5\right]_1^2$$
$$=\pi\cdot\left(\tfrac{1}{5}\cdot 2^5-\tfrac{1}{5}\cdot 1^5\right)=\underline{\underline{\frac{31\pi}{5}}}$$

Oppgave 1.92

a)



Flatestykket er en halvsirkel med sentrum i origo og radius *r*.

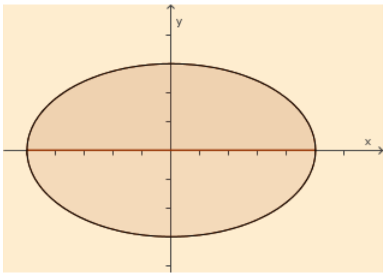
b) Ved å dreie flatestykket 360° om *x*-aksen får man ei kule.

c)

$$V=\pi\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2dx=\pi\int_{-r}^{r}\left(r^2-x^2\right)dx=\pi\cdot\left[r^2x-\tfrac{1}{3}x^3\right]_{-r}^{r}$$
$$=\pi\cdot\left[\left(r^2\cdot r-\tfrac{1}{3}\cdot r^3\right)-\left(r^2\cdot(-r)-\tfrac{1}{3}\cdot(-r)^3\right)\right]=\pi\cdot\left(r^3-\tfrac{1}{3}r^3+r^3-\tfrac{1}{3}r^3\right)=\underline{\underline{\tfrac{4}{3}\pi r^3}}$$

Oppgave 1.93

a)



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \Leftrightarrow \quad \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \quad \Leftrightarrow \quad y^2=b^2-\frac{b^2}{a^2}x^2$$

$$V=\pi\int_{-a}^{a}\left(b^2-\frac{b^2}{a^2}x^2\right)dx=\pi\cdot\left[b^2x-\frac{b^2}{3a^2}x^3\right]_{-a}^{a}$$

$$=\pi\cdot\left[\left(b^2\cdot a-\frac{b^2}{3a^2}\cdot a^3\right)-\left(b^2\cdot(-a)-\frac{b^2}{3a^2}\cdot(-a)^3\right)\right]$$

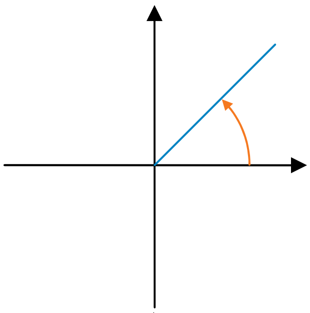
$$=\pi\cdot\left[ab^2-\frac{1}{3}ab^2+ab^2-\frac{1}{3}ab^2\right]=\underline{\underline{\frac{4}{3}\pi ab^2}}$$

b) Når begge halvaksene i ellipsen er like, blir $a=b=r$ og formelen $V=\frac{4}{3}\pi r^3$.

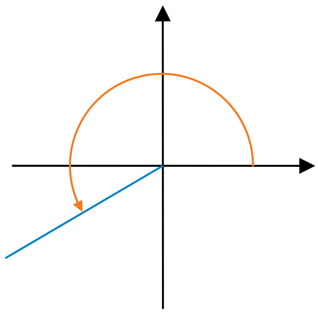
2.1 Vinkler

Oppgave 2.10

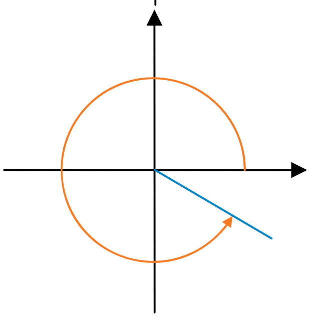
a)



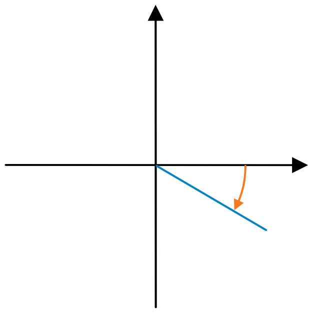
b)



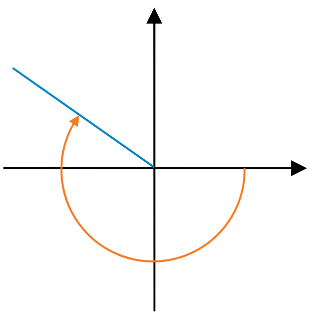
c)



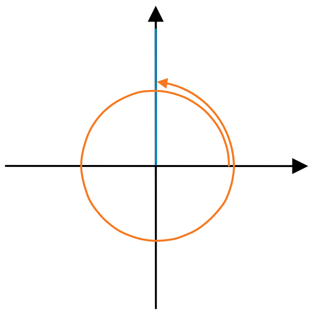
d)



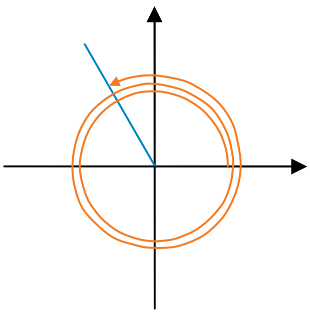
e)



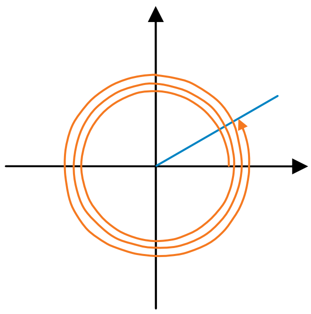
f)



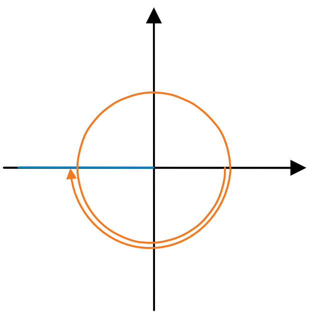
g)



h)

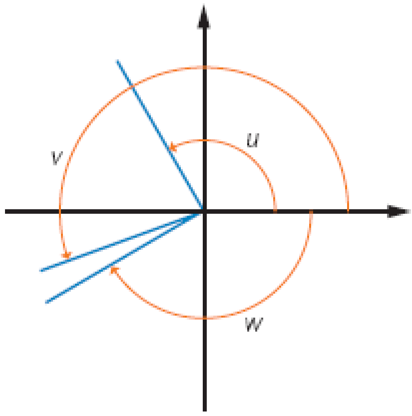


i)



Oppgave 2.11

a)

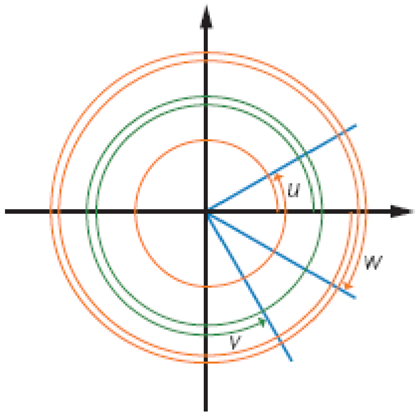


$\underline{\underline{u = 120°}}$

$\underline{\underline{v = 200°}}$

$\underline{\underline{w = -150°}}$

b)



$\underline{\underline{u = 390°}}$

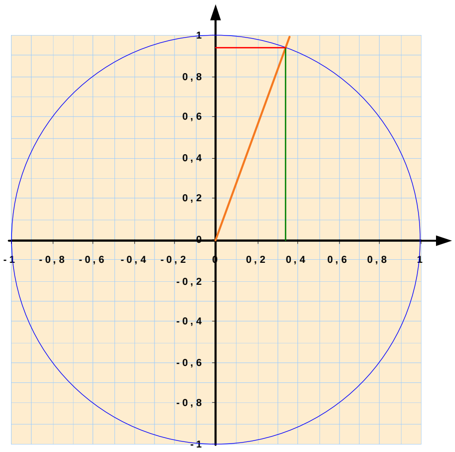
$\underline{\underline{v = 660°}}$

$\underline{\underline{w = -750°}}$

2.2 Generelle trigonometriske definisjoner

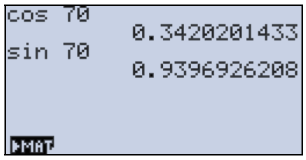
Oppgave 2.20

a)

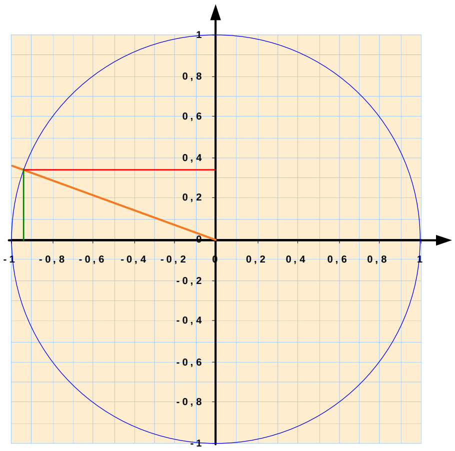


$\underline{\underline{\cos 70° \approx 0,34}}$

$\underline{\underline{\sin 70° \approx 0,94}}$

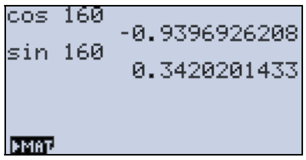


b)

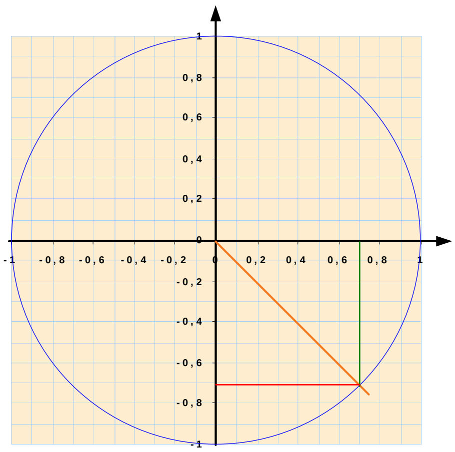


$\underline{\underline{\cos 160° \approx -0,94}}$

$\underline{\underline{\sin 160° \approx 0,34}}$

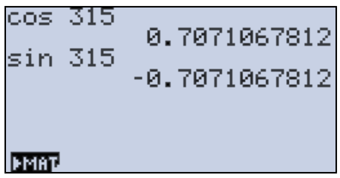


c)

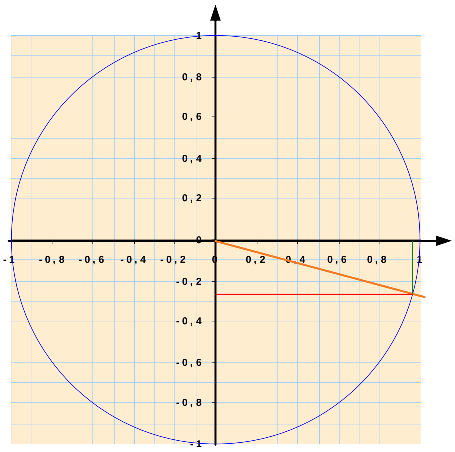


$\underline{\underline{\cos 315° \approx 0,70}}$

$\underline{\underline{\sin 315° \approx -0,70}}$

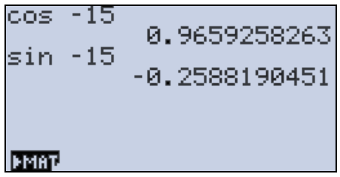


d)



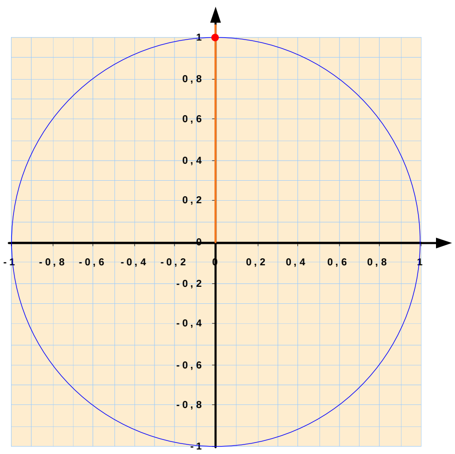
$\underline{\underline{\cos(-15^\circ) \approx 0,95}}$

$\underline{\underline{\sin(-15^\circ) \approx -0,25}}$



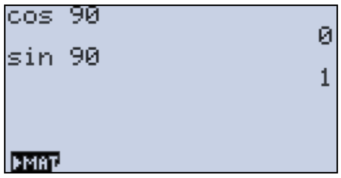
Oppgave 2.21

a)

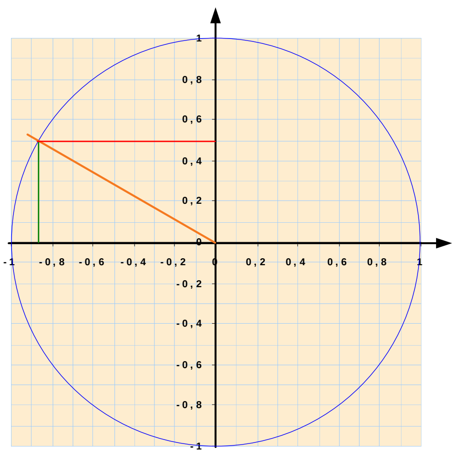


$\underline{\underline{\cos 90^\circ = 0}}$

$\underline{\underline{\sin 90^\circ = 1}}$

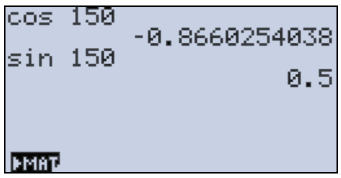


b)

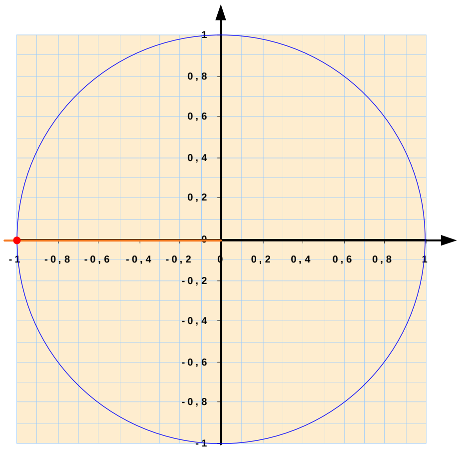


$\underline{\underline{\cos 150^\circ \approx -0,87}}$

$\underline{\underline{\sin 150^\circ = 0,5}}$

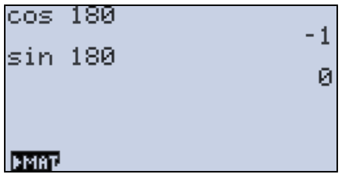


c)

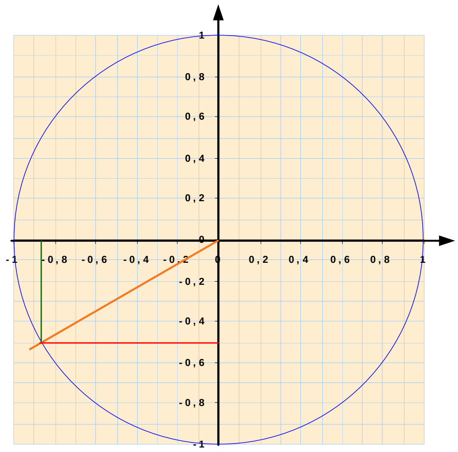


$\underline{\underline{\cos 180° = -1}}$

$\underline{\underline{\sin 180° = 0}}$

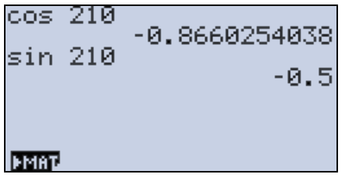


d)



$\underline{\underline{\cos 210° \approx -0{,}87}}$

$\underline{\underline{\sin 210° = -0{,}5}}$



Oppgave 2.22

$$\tan 70° = \frac{\sin 70°}{\cos 70°} \approx \frac{0{,}94}{0{,}34} \approx \underline{\underline{2{,}76}}$$

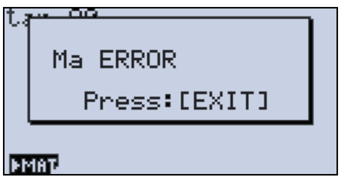
$$\tan 160° = \frac{\sin 160°}{\cos 160°} \approx \frac{0{,}34}{-0{,}94} \approx \underline{\underline{-0{,}36}}$$

$$\tan 315° = \frac{\sin 315°}{\cos 315°} \approx \frac{-0{,}70}{0{,}70} = \underline{\underline{-1}}$$

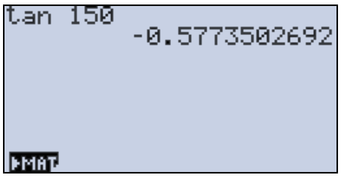
$$\tan(-15°) = \frac{\sin(-15°)}{\cos(-15°)} \approx \frac{-0{,}26}{0{,}97} \approx \underline{\underline{-0{,}27}}$$

Oppgave 2.23

$$\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0}$$ $\underline{\underline{\text{Ikke definert}}}$.

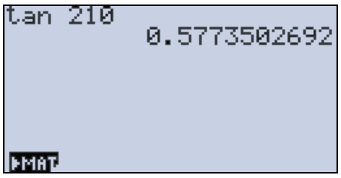


$$\tan 150° = \frac{\sin 150°}{\cos 150°} \approx \frac{0,5}{-0,87} \approx \underline{\underline{-0,57}}$$



$$\tan 180° = \frac{\sin 180°}{\cos 180°} = \frac{0}{-1} = \underline{\underline{0}}$$

$$\tan 210° = \frac{\sin 210°}{\cos 210°} \approx \frac{-0,5}{-0,87} \approx \underline{\underline{0,57}}$$

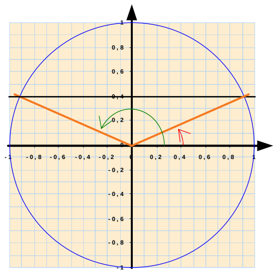


2.3 Trigonometriske likninger

Oppgave 2.30

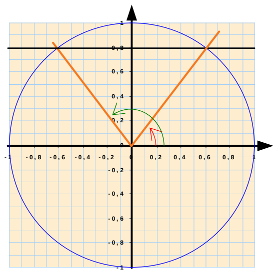
a) $\sin v = 0,4 \qquad x \in [0°, 360°\rangle$

$v = \sin^{-1} 0,4 \approx \underline{\underline{23,6°}} \qquad \vee \qquad v = 180° - 23,6° = \underline{\underline{156,4°}}$



b) $\sin v = 0,8 \qquad x \in [0°, 360°\rangle$

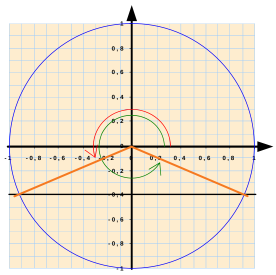
$v = \sin^{-1} 0,8 \approx \underline{\underline{53,1°}} \qquad \vee \qquad v = 180° - 53,1° = \underline{\underline{126,9°}}$



c) $\sin v = -0,4 \qquad x \in [0°, 360°\rangle$

$v = \sin^{-1}(-0,4) \approx -23,6°$ Utenfor intervallet.

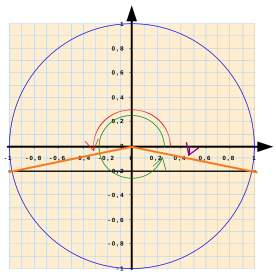
$v = 360° - 23,6° = \underline{\underline{336,4°}} \qquad \vee \qquad v = 180° + 23,6° = \underline{\underline{203,6°}}$



d) $\sin v = -0,2 \qquad x \in [0°, 360°\rangle$

$v = \sin^{-1}(-0,2) \approx -11,5°$ Utenfor intervallet.

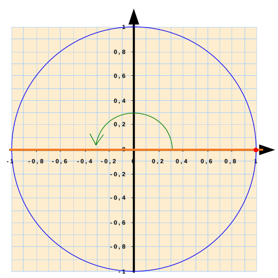
$v = 360° - 11,5° = \underline{\underline{348,5°}} \qquad \vee \qquad v = 180° + 11,5° = \underline{\underline{191,5°}}$



Oppgave 2.31

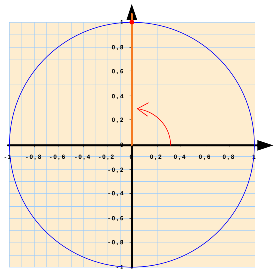
a) $\sin v = 0 \qquad x \in [0°, 360°\rangle$

$v = \sin^{-1} 0 = \underline{\underline{0°}} \qquad \vee \qquad v = 180° - 0° = \underline{\underline{180°}}$



b) $\sin v = 1 \qquad x \in [0°, 360°\rangle$

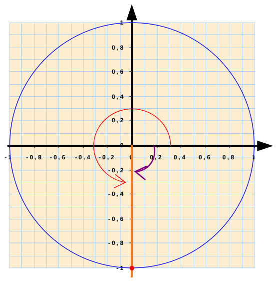
$v = \sin^{-1} 1 = \underline{\underline{90°}}$



c) $\sin v = -1 \qquad x \in [0°, 360°\rangle$

$v = \sin^{-1}(-1) = -90°$ Utenfor intervallet.

$v = 360° - 90° = \underline{\underline{270°}}$



d) $\sin v = 1{,}2 \qquad x \in [0°, 360°\rangle$

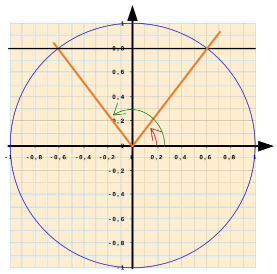
$\underline{\underline{\text{Ingen løsning da } \sin v \in [-1{,}1]}}$

Oppgave 2.32

a) $5\sin v - 4 = 0 \qquad x \in [0°, 360°\rangle$

$\sin v = \frac{4}{5} = 0{,}8$

$v = \sin^{-1} 0{,}8 \approx \underline{\underline{53{,}1°}} \qquad \vee \qquad v = 180° - 53{,}1° = \underline{\underline{126{,}9°}}$

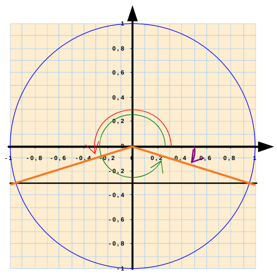


b) $10\sin v + 3 = 0 \qquad x \in [0°, 360°\rangle$

$\sin v = -\frac{3}{10} = -0{,}3$

$v = \sin^{-1}(-0{,}3) = -17{,}5°$ Utenfor intervallet.

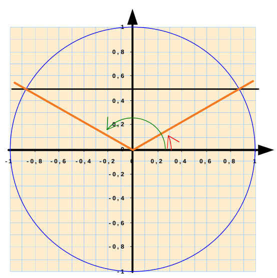
$v = 360° - 17{,}5° = \underline{\underline{342{,}5°}} \qquad \vee \qquad v = 180° + 17{,}5° = \underline{\underline{197{,}5°}}$



c) $2\sin v - 1 = 0 \qquad x \in [0°, 360°\rangle$

$\sin v = \frac{1}{2} = 0{,}5$

$v = \sin^{-1} 0{,}5 = \underline{\underline{30°}} \qquad \vee \qquad v = 180° - 30° = \underline{\underline{150°}}$

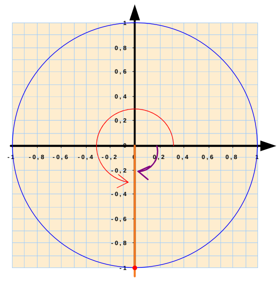


d) $2\sin v+1=-1 \qquad x\in[0°,360°\rangle$

$\sin v=-1$

$v=\sin^{-1}(-1)=-90°$ Utenfor intervallet.

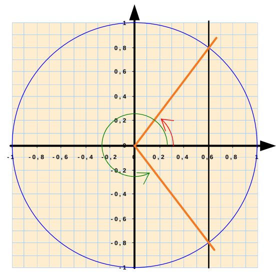
$v=360°-90°=\underline{\underline{270°}}$



Oppgave 2.33

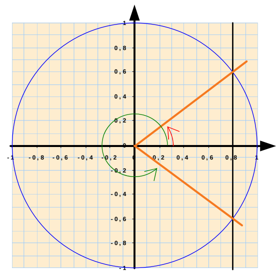
a) $\cos v=0,6 \qquad x\in[0°,360°\rangle$

$v=\cos^{-1}0,6\approx\underline{\underline{53,1°}} \quad \vee \quad v=360°-53,1°=\underline{\underline{306,9°}}$



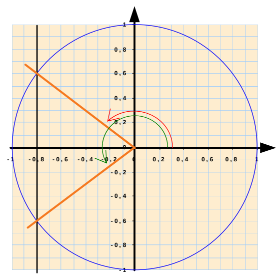
b) $\cos v=0,8 \qquad x\in[0°,360°\rangle$

$v=\cos^{-1}0,8\approx\underline{\underline{36,9°}} \quad \vee \quad v=360°-36,9°=\underline{\underline{323,1°}}$



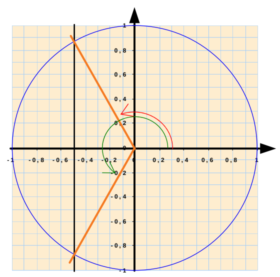
c) $\cos v=-0,8 \qquad x\in[0°,360°\rangle$

$v=\cos^{-1}(-0,8)\approx\underline{\underline{143,1°}} \quad \vee \quad v=360°-143,1°=\underline{\underline{216,9°}}$



d) $\cos v=-0,5 \qquad x\in[0°,360°\rangle$

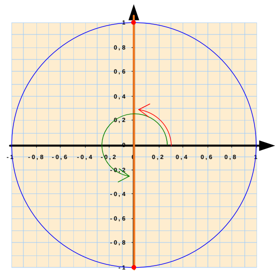
$v=\cos^{-1}(-0,5)=\underline{\underline{120°}} \quad \vee \quad v=360°-120°=\underline{\underline{240°}}$



Oppgave 2.34

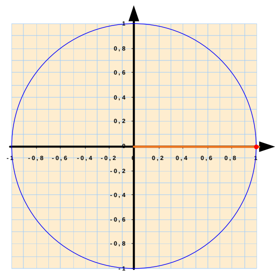
a) $\cos v = 0 \qquad x \in [0°, 360°\rangle$

$v = \cos^{-1} 0 = \underline{\underline{90°}} \qquad \vee \qquad v = 360° - 90° = \underline{\underline{270°}}$



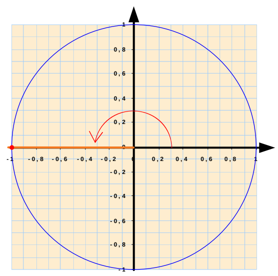
b) $\cos v = 1 \qquad x \in [0°, 360°\rangle$

$v = \cos^{-1} 1 = \underline{\underline{0°}}$



c) $\cos v = -1 \qquad x \in [0°, 360°\rangle$

$v = \cos^{-1}(-1) = \underline{\underline{180°}}$



d) $\cos v = -1{,}1 \qquad x \in [0°, 360°\rangle$

$\underline{\underline{\text{Ingen løsning da } \cos v \in [-1{,}1]}}$

Oppgave 2.35

a) $\tan v = 1 \qquad x \in [0°, 360°\rangle$

$$v = \tan^{-1} 1 = 45° + n \cdot 180° \qquad \begin{cases} n = 0 \Rightarrow \underline{\underline{v = 45°}} \\ n = 1 \Rightarrow \underline{\underline{v = 225°}} \end{cases}$$

b) $\tan v = -2 \qquad x \in [0°, 360°\rangle$

$$v = \tan^{-1}(-2) \approx -63{,}4° + n \cdot 180° \qquad \begin{cases} n = 1 \Rightarrow \underline{\underline{v = 116{,}6°}} \\ n = 2 \Rightarrow \underline{\underline{v = 296{,}6°}} \end{cases}$$

c) $5 \tan v + 8 = 0 \qquad x \in [0°, 360°\rangle$

$\tan v = -\frac{8}{5} = -1{,}6$

$$v = \tan^{-1}(-1{,}6) = -58{,}0° + n \cdot 180° \qquad \begin{cases} n = 1 \Rightarrow \underline{\underline{v = 122{,}0°}} \\ n = 2 \Rightarrow \underline{\underline{v = 302{,}0°}} \end{cases}$$

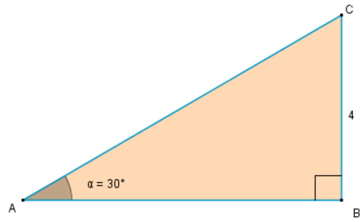
d) $3\tan v + 2 = 2(4 - \tan v) \qquad x \in [0°, 360°\rangle$

$3\tan v + 2\tan v = 8 - 2 \quad \Leftrightarrow \quad 5\tan v = 6 \quad \Leftrightarrow \quad \tan v = \frac{6}{5} = 1,2$

$$v = \tan^{-1} 1,2 \approx 50,2° + n \cdot 180° \qquad \begin{cases} n = 0 \quad \Rightarrow \quad \underline{\underline{v = 50,2°}} \\ n = 1 \quad \Rightarrow \quad \underline{\underline{v = 230,2°}} \end{cases}$$

2.4 Eksakte trigonometriske verdier

Oppgave 2.40

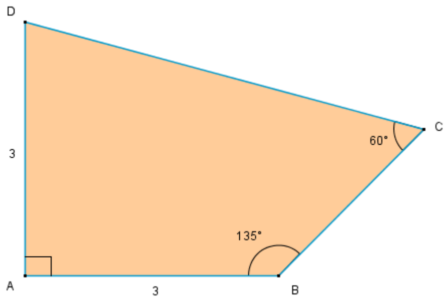


$$\tan 30° = \frac{4}{AB} \quad \Leftrightarrow \quad AB = \frac{4}{\tan 30°} = \frac{4}{\frac{1}{\sqrt{3}}} = \underline{\underline{4\sqrt{3}}}$$

$$\sin 30° = \frac{4}{AC} \quad \Leftrightarrow \quad AC = \frac{4}{\sin 30°} = \frac{4}{\frac{1}{2}} = \underline{\underline{8}}$$

Oppgave 2.41

a)



$$BD = \sqrt{3^2 + 3^2} = \sqrt{2 \cdot 3^2} = \sqrt{2} \cdot \sqrt{3^2} = \underline{\underline{3\sqrt{2}}}$$

b) ΔABD er likebeint og derfor er $\angle ABD = \angle ADB = 45°$

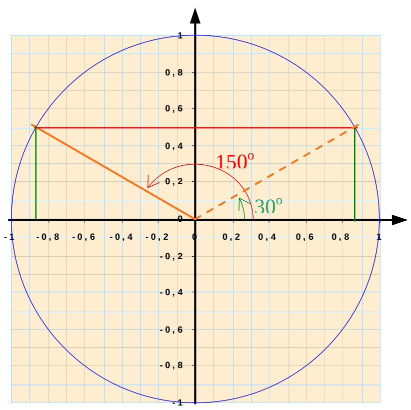
$\Rightarrow \quad \angle CBD = 135° - 45° = 90° \quad \Rightarrow \quad \Delta BCD$ er rettvinklet.

$$\tan 60° = \frac{3\sqrt{2}}{BC} \quad \Leftrightarrow \quad BC = \frac{3\sqrt{2}}{\tan 60°} = \frac{3\sqrt{2}}{\sqrt{3}} = \frac{3\sqrt{2} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \sqrt{2} \cdot \sqrt{3} = \underline{\underline{\sqrt{6}}}$$

$$\cos 60° = \frac{\sqrt{6}}{DC} \quad \Leftrightarrow \quad DC = \frac{\sqrt{6}}{\cos 60°} = \frac{\sqrt{6}}{\frac{1}{2}} = \underline{\underline{2\sqrt{6}}}$$

Oppgave 2.42

a)



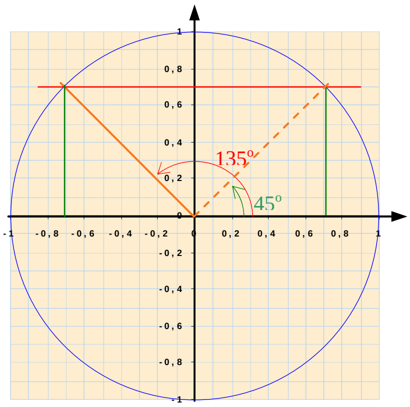
På grunn av symmetri får vi:

$$\sin 150° = \sin 30° = \underline{\underline{\frac{1}{2}}}$$

$$\cos 150° = -\cos 30° = \underline{\underline{-\frac{\sqrt{3}}{2}}}$$

$$\tan 150° = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \underline{\underline{-\frac{1}{\sqrt{3}}}}$$

b)



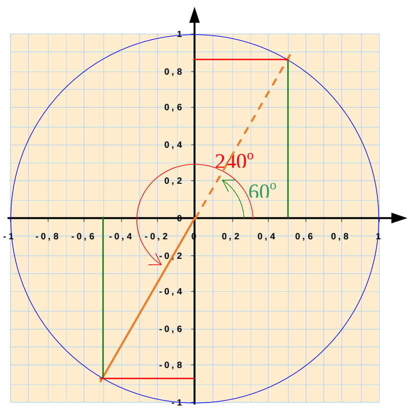
På grunn av symmetri får vi:

$$\sin 135° = \sin 45° = \underline{\underline{\frac{\sqrt{2}}{2}}}$$

$$\cos 135° = -\cos 45° = \underline{\underline{-\frac{\sqrt{2}}{2}}}$$

$$\tan 135° = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \underline{\underline{-1}}$$

c)



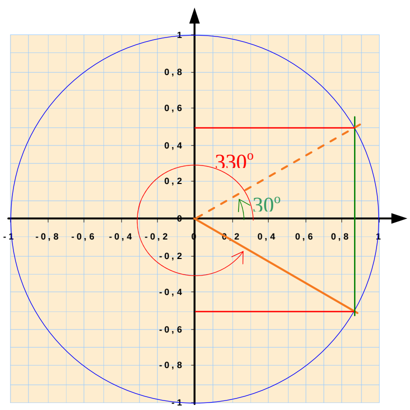
På grunn av symmetri får vi:

$$\sin 240° = -\sin 60° = \underline{\underline{-\frac{\sqrt{3}}{2}}}$$

$$\cos 240° = -\cos 60° = \underline{\underline{-\frac{1}{2}}}$$

$$\tan 240° = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \underline{\underline{\sqrt{3}}}$$

d)



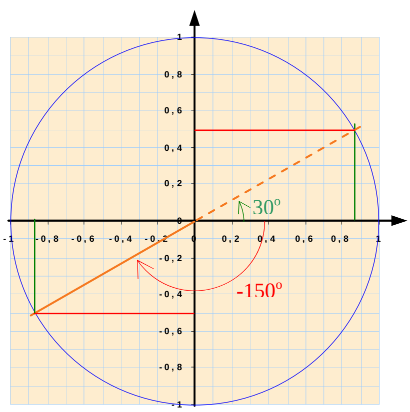
På grunn av symmetri får vi:

$$\sin 330° = -\sin 30° = \underline{\underline{-\frac{1}{2}}}$$

$$\cos 330° = \cos 30° = \underline{\underline{\frac{\sqrt{3}}{2}}}$$

$$\tan 330° = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \underline{\underline{-\frac{1}{\sqrt{3}}}}$$

e)



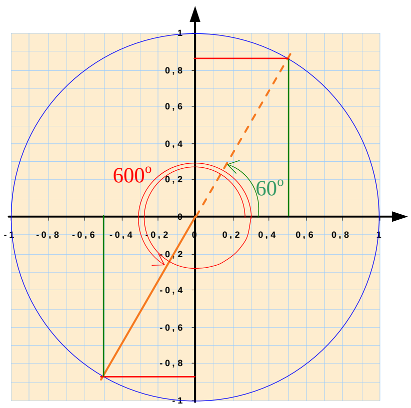
På grunn av symmetri får vi:

$$\sin(-150°) = -\sin 30° = \underline{\underline{-\frac{1}{2}}}$$

$$\cos(-150°) = -\cos 30° = \underline{\underline{-\frac{\sqrt{3}}{2}}}$$

$$\tan(-150°) = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \underline{\underline{\frac{1}{\sqrt{3}}}}$$

f)



På grunn av symmetri får vi:

$$\sin 600° = -\sin 60° = \underline{\underline{-\frac{\sqrt{3}}{2}}}$$

$$\cos 600° = -\cos 60° = \underline{\underline{-\frac{1}{2}}}$$

$$\tan 600° = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \underline{\underline{\sqrt{3}}}$$

2.5 Absolutt vinkelmål

Oppgave 2.50

a) $30° = \frac{180°}{6} = \underline{\underline{\frac{\pi}{6}}}$

b) $45° = \frac{180°}{4} = \underline{\underline{\frac{\pi}{4}}}$

c) $90° = \frac{180°}{2} = \underline{\underline{\frac{\pi}{2}}}$

d) $360° = 2 \cdot 180° = \underline{\underline{2\pi}}$

e) $120° = \frac{2}{3} \cdot 180° = \underline{\underline{\frac{2\pi}{3}}}$

Oppgave 2.51

a) $\frac{\pi}{6} = \frac{180°}{6} = \underline{\underline{30°}}$

b) $\frac{3\pi}{4} = \frac{3 \cdot 180°}{4} = \underline{\underline{135°}}$

c) $\frac{5\pi}{6} = \frac{5 \cdot 180°}{6} = \underline{\underline{150°}}$

d) $\frac{3\pi}{2} = \frac{3 \cdot 180°}{2} = \underline{\underline{270°}}$

Oppgave 2.52

a) $135° = \frac{135°}{180°} \cdot \pi = \underline{\underline{\frac{3\pi}{4}}}$

b) $150° = \frac{150°}{180°} \cdot \pi = \underline{\underline{\frac{5\pi}{6}}}$

c) $240° = \frac{240°}{180°} \cdot \pi = \underline{\underline{\frac{4\pi}{3}}}$

d) $720° = \frac{720°}{180°} \cdot \pi = \underline{\underline{4\pi}}$

e) $112° = \frac{112°}{180°} \cdot \pi = \underline{\underline{\frac{28\pi}{45}}} \approx \underline{\underline{1{,}95}}$

f) $237{,}5° = \frac{237{,}5°}{180°} \cdot \pi \approx \underline{\underline{4{,}15}}$

Oppgave 2.53

a) $\frac{2\pi}{3} = \frac{2\pi}{3} \cdot \frac{180°}{\pi} = \underline{\underline{120°}}$

b) $\frac{7\pi}{4} = \frac{7\pi}{4} \cdot \frac{180°}{\pi} = \underline{\underline{315°}}$

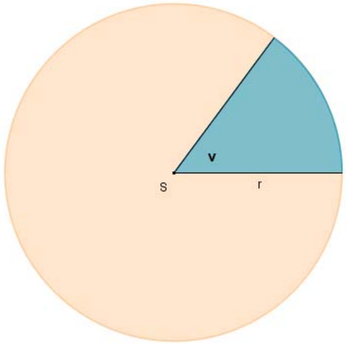
c) $\frac{11\pi}{6} = \frac{11\pi}{6} \cdot \frac{180°}{\pi} = \underline{\underline{330°}}$

d) $1 = 1 \cdot \frac{180°}{\pi} \approx \underline{\underline{57{,}3°}}$

e) $2{,}45 = 2{,}45 \cdot \frac{180°}{\pi} \approx \underline{\underline{140{,}4°}}$

f) $-7{,}6 = -7{,}6 \cdot \frac{180°}{\pi} \approx \underline{\underline{-435{,}4°}}$

Oppgave 2.54



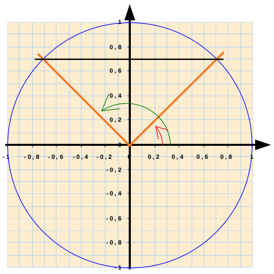
$$A = \pi r^2 \cdot \frac{v}{2\pi} = \underline{\underline{\frac{1}{2} v r^2}}$$

2.6 Trigonometriske likninger og radianer

Oppgave 2.60

a) $\sin x = 0,7 \qquad x \in [0, 2\pi\rangle$

$x = \sin^{-1} 0,7 \approx \underline{\underline{0,78}} \qquad \vee \qquad x = \pi - 0,78 \approx \underline{\underline{2,36}}$

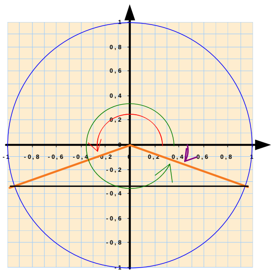


b) $3\sin x + 2 = 1 \qquad x \in [0, 2\pi\rangle$

$\sin x = \frac{1-2}{3} \quad \Leftrightarrow \quad \sin x = -\frac{1}{3} \quad \Leftrightarrow$

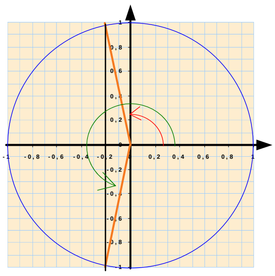
$v = \sin^{-1}\left(-\frac{1}{3}\right) \approx -0,34$ Utenfor intervallet.

$v = 2\pi - 0,34 \approx \underline{\underline{5,94}} \qquad \vee \qquad v = \pi + 0,34 \approx \underline{\underline{3,48}}$



c) $\cos x = -0,2 \qquad x \in [0, 2\pi\rangle$

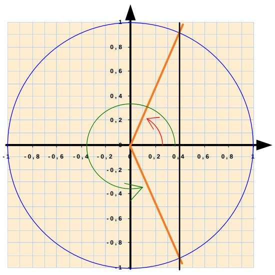
$x = \cos^{-1}(-0,2) \approx \underline{\underline{1,77}} \qquad \vee \qquad x = 2\pi - 1,77 \approx \underline{\underline{4,51}}$



d) $5\cos x - 2 = 0 \qquad x \in [0, 2\pi\rangle$

$\cos x = \frac{2}{5} = 0,4$

$x = \cos^{-1}(0,4) \approx \underline{\underline{1,16}} \qquad \vee \qquad x = 2\pi - 1,16 \approx \underline{\underline{5,12}}$



Oppgave 2.61

a) $2\tan x = 1 \qquad x \in [-\pi, \pi\rangle$

$\tan x = \frac{1}{2} = 0,5$

$$x = \tan^{-1} 0,5 \approx 0,46 + n \cdot \pi \qquad \begin{cases} n = -1 \ \Rightarrow \ \underline{\underline{x = -2,68}} \\ n = 0 \ \Rightarrow \ \underline{\underline{x = 0,46}} \end{cases}$$

b) $3\tan x + 4 = 0 \qquad x \in [-\pi, \pi\rangle$

$\tan x = -\frac{4}{3}$

$$x = \tan^{-1}\left(-\tfrac{4}{3}\right) \approx -0{,}93 + n\cdot\pi \qquad \begin{cases} n = 0 \;\Rightarrow\; \underline{\underline{x = -0{,}93}} \\ n = 1 \;\Rightarrow\; \underline{\underline{x = 2{,}21}} \end{cases}$$

c) $2\sin x - 3\cos x = 0 \qquad x \in [-\pi, \pi\rangle$

$$\overset{\cos x \neq 0}{\Leftrightarrow} \quad 2\tan x - 3 = 0 \;\Leftrightarrow\; \tan x = \tfrac{3}{2}$$

$$x = \tan^{-1}\left(\tfrac{3}{2}\right) \approx 0{,}98 + n\cdot\pi \qquad \begin{cases} n = -1 \;\Rightarrow\; \underline{\underline{x = -2{,}16}} \\ n = 0 \;\Rightarrow\; \underline{\underline{x = 0{,}98}} \end{cases}$$

d) $3\sin x + 2\cos x = 0 \qquad x \in [-\pi, \pi\rangle$

$$\overset{\cos x \neq 0}{\Leftrightarrow} \quad 3\tan x + 2 = 0 \;\Leftrightarrow\; \tan x = -\tfrac{2}{3}$$

$$x = \tan^{-1}\left(-\tfrac{2}{3}\right) \approx -0{,}59 + n\cdot\pi \qquad \begin{cases} n = 0 \;\Rightarrow\; \underline{\underline{x = -0{,}59}} \\ n = 1 \;\Rightarrow\; \underline{\underline{x = 2{,}55}} \end{cases}$$

Oppgave 2.62

a) $10\sin^2 x - 5\sin x + 0{,}6 = 0 \qquad x \in [0, 2\pi]$

$$\sin x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\cdot 10\cdot 0{,}6}}{2\cdot 10} = \frac{5 \pm \sqrt{1}}{20}$$

$$\Leftrightarrow \begin{cases} \sin x = \frac{3}{10} \\ \vee \\ \sin x = \frac{1}{5} \end{cases} \Leftrightarrow \begin{cases} x = \sin^{-1}\left(\frac{3}{10}\right) \approx \underline{\underline{0{,}30}} \;\vee\; x \approx \pi - 0{,}30 \approx \underline{\underline{2{,}84}} \\ \vee \\ x = \sin^{-1}\left(\frac{1}{5}\right) \approx \underline{\underline{0{,}20}} \;\vee\; x = \pi - 0{,}20 \approx \underline{\underline{2{,}94}} \end{cases}$$

b) $3\cos^2 x - 8\cos x - 3 = 0 \qquad x \in [0, 2\pi]$

$$\cos x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4\cdot 3\cdot(-3)}}{2\cdot 3} = \frac{8 \pm \sqrt{100}}{6}$$

$$\Leftrightarrow \begin{cases} \cos x = 3 \quad \text{umulig} \\ \vee \\ \cos x = -\frac{1}{3} \end{cases} \Leftrightarrow \quad x = \cos^{1}\left(-\tfrac{1}{3}\right) \approx \underline{\underline{1{,}91}} \;\vee\; x = 2\pi - 1{,}91 \approx \underline{\underline{4{,}37}}$$

c) $\tan^2 x - 5\tan x + 6 = 0 \qquad x \in [0, 2\pi]$

$$\cos x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\cdot 1\cdot 6}}{2\cdot 1} = \frac{5 \pm \sqrt{1}}{2}$$

$$\Leftrightarrow \begin{cases} \tan x = 3 \\ \vee \\ \tan x = 2 \end{cases} \Leftrightarrow \begin{cases} x = \tan^{-1} 3 \approx 1,25 + n\cdot\pi \\ \vee \\ x = \tan^{-1} 2 \approx 1,11 + n\cdot\pi \end{cases} \Leftrightarrow \begin{cases} \underline{\underline{x = 1,25}} \quad \vee \quad \underline{\underline{x = 4,39}} \\ \vee \\ \underline{\underline{x = 1,11}} \quad \vee \quad \underline{\underline{x = 4,25}} \end{cases}$$

Oppgave 2.63

$\sin^2 x - 3\sin x \cdot \cos x + 2\cos^2 x = 0 \qquad x \in [-\pi, \pi]$

$$\overset{\cos^2 x \neq 0}{\Leftrightarrow} \frac{\sin^2 x}{\cos^2 x} - \frac{3\sin x\cdot\cos x}{\cos^2 x} + \frac{2\cos^2 x}{\cos^2 x} = 0 \Leftrightarrow \tan^2 x - 3\tan x + 2 = 0 \Leftrightarrow$$

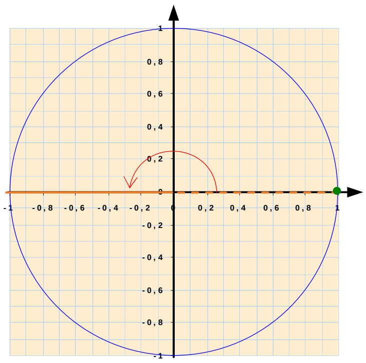
$$\tan x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\cdot 1\cdot 2}}{2\cdot 1} = \frac{3 \pm \sqrt{1}}{2}$$

$$\Leftrightarrow \begin{cases} \tan x = 2 \\ \vee \\ \tan x = 1 \end{cases} \Leftrightarrow \begin{cases} x = \tan^{-1} 2 \approx 1,11 + n\cdot\pi \\ \vee \\ x = \tan^{-1} 1 = \frac{\pi}{4} + n\cdot\pi \end{cases} \Leftrightarrow \begin{cases} \underline{\underline{x = -2,03}} \quad \vee \quad \underline{\underline{x = 1,11}} \\ \vee \\ \underline{\underline{x = -\frac{3\pi}{4}}} \quad \vee \quad \underline{\underline{x = \frac{\pi}{4}}} \end{cases}$$

2.7 Eksakte løsninger

Oppgave 2.70

a)

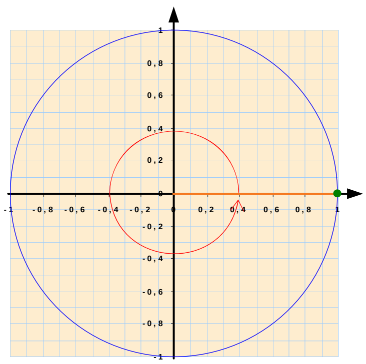


$$\cos \pi = -\cos 0 = \underline{\underline{-1}}$$

$$\sin \pi = \sin 0 = \underline{\underline{0}}$$

$$\tan \pi = \frac{\sin \pi}{\cos \pi} = \frac{0}{-1} = \underline{\underline{0}}$$

b)

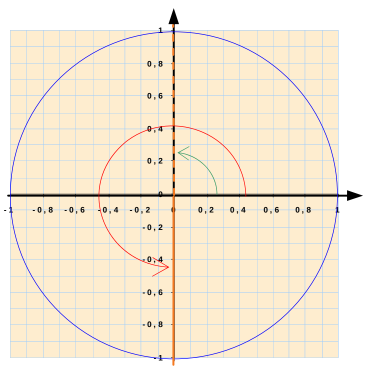


$$\cos 2\pi = \cos 0 = \underline{\underline{1}}$$

$$\sin 2\pi = \sin 0 = \underline{\underline{0}}$$

$$\tan 2\pi = \frac{\sin 2\pi}{\cos 2\pi} = \frac{0}{1} = \underline{\underline{0}}$$

c)

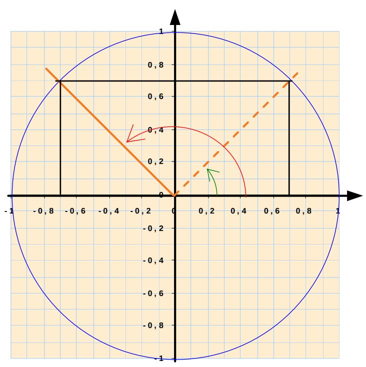


$$\cos \frac{3\pi}{2} = \cos \frac{\pi}{2} = \underline{\underline{0}}$$

$$\sin \frac{3\pi}{2} = -\sin \frac{\pi}{2} = \underline{\underline{-1}}$$

$$\tan \frac{3\pi}{2} = \frac{\sin \frac{3\pi}{2}}{\cos \frac{3\pi}{2}} = \frac{-1}{0} \quad \underline{\underline{\text{Ikke definert}}}.$$

d)

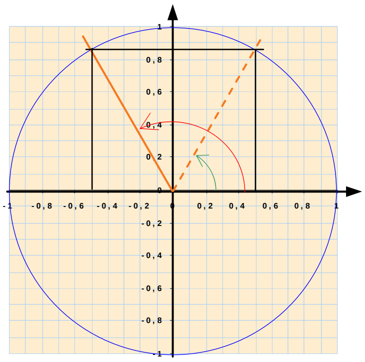


$$\cos \frac{3\pi}{4} = -\cos\left(\pi - \frac{3\pi}{4}\right) = -\cos \frac{\pi}{4} = \underline{\underline{-\frac{\sqrt{2}}{2}}}$$

$$\sin \frac{3\pi}{4} = \sin\left(\pi - \frac{3\pi}{4}\right) = \sin \frac{\pi}{4} = \underline{\underline{\frac{\sqrt{2}}{2}}}$$

$$\tan \frac{3\pi}{4} = \frac{\sin \frac{3\pi}{4}}{\cos \frac{3\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \underline{\underline{-1}}$$

e)

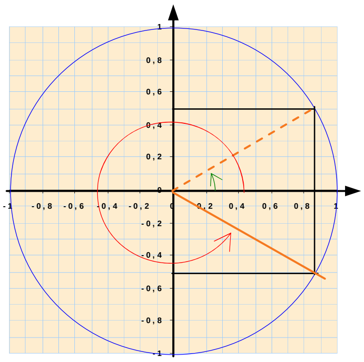


$$\cos\frac{2\pi}{3} = -\cos\left(\pi - \frac{2\pi}{3}\right) = -\cos\frac{\pi}{3} = \underline{\underline{-\frac{1}{2}}}$$

$$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{2\pi}{3}\right) = \sin\frac{\pi}{3} = \underline{\underline{\frac{\sqrt{3}}{2}}}$$

$$\tan\frac{2\pi}{3} = \frac{\sin\frac{2\pi}{3}}{\cos\frac{2\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \underline{\underline{-\sqrt{3}}}$$

f)



$$\cos\frac{11\pi}{6} = \cos\left(2\pi - \frac{11\pi}{6}\right) = \cos\frac{\pi}{6} = \underline{\underline{\frac{\sqrt{3}}{2}}}$$

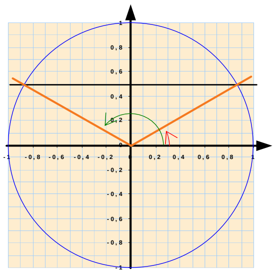
$$\sin\frac{11\pi}{6} = -\sin\left(2\pi - \frac{11\pi}{6}\right) = -\sin\frac{\pi}{6} = \underline{\underline{-\frac{1}{2}}}$$

$$\tan\frac{11\pi}{6} = \frac{\sin\frac{11\pi}{6}}{\cos\frac{11\pi}{6}} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \underline{\underline{-\frac{1}{\sqrt{3}}}}$$

Oppgave 2.71

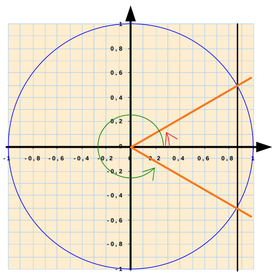
a) $4\sin x - 1 = 1 \quad x \in [0, 2\pi\rangle$

$$\sin x = \frac{1+1}{4} \Leftrightarrow \sin x = \frac{1}{2} \Leftrightarrow \underline{\underline{x = \frac{\pi}{6}}} \vee x = \pi - \frac{\pi}{6} = \underline{\underline{\frac{5\pi}{6}}}$$



b) $2\cos x - \sqrt{3} = 0 \quad x \in [0, 2\pi\rangle$

$$\cos x = \frac{\sqrt{3}}{2} \Leftrightarrow \underline{\underline{x = \frac{\pi}{6}}} \vee x = 2\pi - \frac{\pi}{6} = \underline{\underline{\frac{11\pi}{6}}}$$

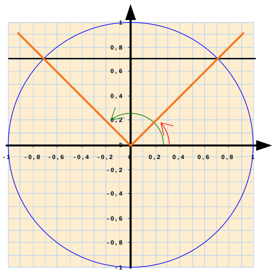


c) $3\tan x + 3 = 0 \quad x \in [0, 2\pi\rangle$

$$\tan x = \frac{-3}{3} = -1 \Leftrightarrow x = -\frac{\pi}{4} + n\cdot\pi \Leftrightarrow \underline{\underline{x = \frac{3\pi}{4}}} \vee \underline{\underline{x = \frac{7\pi}{4}}}$$

Oppgave 2.72

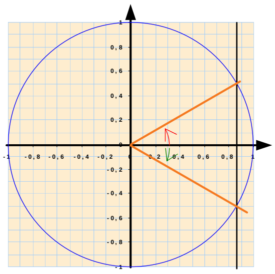
a) $4\sin(\pi x) = 2\sqrt{2} \qquad x \in [-1,1]$



$\sin(\pi x) = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} \quad \Leftrightarrow \quad \pi x = \frac{\pi}{4} \quad \vee \quad \pi x = \pi - \frac{\pi}{4} \quad \Leftrightarrow$

$\pi x = \frac{\pi}{4} \quad \vee \quad \pi x = \frac{3\pi}{4} \quad \overset{:\pi}{\Leftrightarrow} \quad \underline{\underline{x = \frac{1}{4}}} \quad \vee \quad \underline{\underline{x = \frac{3}{4}}}$

b) $2\cos\left(\frac{\pi}{2}x\right) = \sqrt{3} \qquad x \in [-2,2]$



$\cos\left(\frac{\pi}{2}x\right) = \frac{\sqrt{3}}{2} \quad \Leftrightarrow \quad \frac{\pi}{2}x = \frac{\pi}{6} \quad \vee \quad \frac{\pi}{2}x = -\frac{\pi}{6} \quad \overset{:\frac{\pi}{2}}{\Leftrightarrow}$

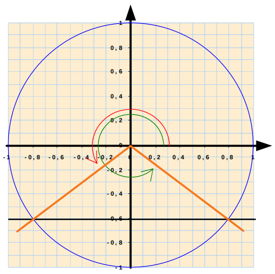
$\underline{\underline{x = \frac{1}{3}}} \quad \vee \quad \underline{\underline{x = -\frac{1}{3}}}$

c) $\tan(\pi x) - \sqrt{3} = 0 \qquad x \in [0,2]$

$\tan(\pi x) = \sqrt{3} \quad \Leftrightarrow \quad \pi x = \frac{\pi}{3} + n \cdot \pi \quad \overset{:\pi}{\Leftrightarrow} \quad x = \frac{1}{3} + n \quad \Leftrightarrow$

$\underline{\underline{x = \frac{1}{3}}} \quad \vee \quad \underline{\underline{x = \frac{4}{3}}}$

d) $5\sin\left(\frac{\pi}{5}x\right) + 3 = 0 \qquad x \in [0,10]$



$\sin\left(\frac{\pi}{5}x\right) = -\frac{3}{5} \quad \Leftrightarrow \quad \frac{\pi}{5}x \approx -0,64 \quad \vee \quad \frac{\pi}{5}x \approx \pi + 0,64 \quad \Leftrightarrow$

$\frac{\pi}{5}x \approx 2\pi - 0,64 \quad \vee \quad \frac{\pi}{5}x \approx \pi + 0,64 \quad \Leftrightarrow$

$\frac{\pi}{5}x \approx 5,64 \quad \vee \quad \frac{\pi}{5}x \approx 3,78 \quad \overset{:\frac{\pi}{5}}{\Leftrightarrow} \quad \underline{\underline{x = 9,0}} \quad \vee \quad \underline{\underline{x = 6,0}}$

Oppgave 2.73

a) $2\sin^2 x + \sin x - 1 = 0 \qquad x \in [0, 2\pi]$

$$\sin x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{-1 \pm \sqrt{9}}{4}$$

$$\Leftrightarrow \begin{cases} \sin x = \frac{1}{2} \\ \vee \\ \sin x = -1 \end{cases} \quad \Leftrightarrow \begin{cases} x = \sin^{-1}\left(\frac{1}{2}\right) = \underline{\underline{\frac{\pi}{6}}} \quad \vee \quad x = \pi - \frac{\pi}{6} = \underline{\underline{\frac{5\pi}{6}}} \\ \vee \\ x = \sin^{-1}(-1) = \underline{\underline{\frac{3\pi}{2}}} \end{cases}$$

b) $2\cos^2 x+\sqrt{2}\cos x-2=0 \qquad x\in[0,2\pi]$

$$\cos x=\frac{-\sqrt{2}\pm\sqrt{\left(\sqrt{2}\right)^2-4\cdot 2\cdot(-2)}}{2\cdot 2}=\frac{-\sqrt{2}\pm\sqrt{18}}{4}=\frac{-\sqrt{2}\pm 3\cdot\sqrt{2}}{4}$$

$$\Leftrightarrow \begin{cases}\cos x=\frac{2\sqrt{2}}{4}=\frac{\sqrt{2}}{2}\\ \vee\\ \sin x=-\sqrt{2} \quad \text{umulig}\end{cases} \quad \Leftrightarrow \quad x=\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)=\underline{\underline{\frac{\pi}{4}}} \quad \vee \quad x=2\pi-\frac{\pi}{4}=\underline{\underline{\frac{7\pi}{4}}}$$

2.8 Enhetsformelen

Oppgave 2.80

a) $\sin v = \frac{4}{5} \quad v \in [90°, 180°]$

$\sin^2 v + \cos^2 v = 1 \quad \Rightarrow$

$$\left(\frac{4}{5}\right)^2 + \cos^2 v = 1 \quad \Leftrightarrow \quad \cos v = \pm\sqrt{1-\left(\frac{4}{5}\right)^2} = \pm\sqrt{1-\frac{16}{25}} = \pm\sqrt{\frac{9}{25}} \overset{v \text{ i 2.kvadrant}}{\Rightarrow} \underline{\underline{\cos v = -\frac{3}{5}}}$$

$$\tan v = \frac{\sin v}{\cos v} = \frac{\frac{4}{5}}{-\frac{3}{5}} = \underline{\underline{-\frac{4}{3}}}$$

b) $\sin v = -\frac{1}{4} \quad v \in \left[\frac{3\pi}{2}, 2\pi\right]$

$\sin^2 v + \cos^2 v = 1 \quad \Rightarrow$

$$\cos v = \pm\sqrt{1-\left(-\frac{1}{4}\right)^2} = \pm\sqrt{1-\frac{1}{16}} = \pm\sqrt{\frac{15}{16}} = \pm\frac{\sqrt{15}}{\sqrt{16}} = \pm\frac{\sqrt{15}}{4} \overset{v \text{ i 4.kvadrant}}{\Rightarrow} \underline{\underline{\cos v = \frac{\sqrt{15}}{4}}}$$

$$\tan v = \frac{\sin v}{\cos v} = \frac{-\frac{1}{4}}{\frac{\sqrt{15}}{4}} = \underline{\underline{-\frac{1}{\sqrt{15}}}}$$

Oppgave 2.81

$\tan v = 2 \quad v \in [0°, 90°]$

$$\Leftrightarrow \frac{\sin v}{\cos v} = 2 \quad \Leftrightarrow \quad \sin v = 2\cos v$$

$$\sin^2 v + \cos^2 v = 1 \quad \Rightarrow \quad (2\cos v)^2 + \quad \Leftrightarrow \quad 4\cos^2 v + \cos^2 v = 1 \quad \Leftrightarrow \quad 5\cos^2 v = 1$$

$$\Leftrightarrow \quad \cos^2 v = \frac{1}{5} \quad \Leftrightarrow \quad \cos v = \pm\sqrt{\frac{1}{5}} = \pm\frac{1}{\sqrt{5}} \overset{v \text{ i 1.kvadrant}}{\Rightarrow} \underline{\underline{\cos v = \frac{1}{\sqrt{5}}}}$$

$$\sin v = 2\cos v \Rightarrow \quad \underline{\underline{\sin v = \frac{2}{\sqrt{5}}}}$$

Oppgave 2.82

a) $4\sin^2 x - \sin x\cdot\cos x + \cos^2 x = 3 \qquad x\in[0,2\pi]$

$\Leftrightarrow\ 4\sin^2 x - \sin x\cdot\cos x + \cos^2 x = 3\cdot\left(\sin^2 x + \cos^2 x\right)$

$\Leftrightarrow\ 4\sin^2 x - \sin x\cdot\cos x + \cos^2 x = 3\sin^2 x + 3\cos^2 x$

$\Leftrightarrow\ 4\sin^2 x - \sin x\cdot\cos x + \cos^2 x - 3\sin^2 x - 3\cos^2 x = 0$

$\Leftrightarrow\ \sin^2 x - \sin x\cdot\cos x - 2\cos^2 x = 0 \quad \overset{\cos^2 x\neq 0}{\Leftrightarrow} \quad \frac{\sin^2 x}{\cos^2 x} - \frac{\sin x\cdot\cos x}{\cos^2 x} - \frac{2\cos^2 x}{\cos^2 x} = \frac{0}{\cos^2 x}$

$\Leftrightarrow\ \tan^2 x - \tan x - 2 = 0 \quad \Leftrightarrow \quad \tan x = \frac{-(-1)\pm\sqrt{(-1)^2 - 4\cdot 1\cdot(-2)}}{2\cdot 1} = \frac{1\pm\sqrt{9}}{2}$

$$\Leftrightarrow \begin{cases}\tan x = 2\\ \vee\\ \tan x = -1\end{cases} \Leftrightarrow \begin{cases}x = \tan^{-1} 2\\ \vee\\ x = \tan^{-1}(-1)\end{cases} \Leftrightarrow \begin{cases}x \approx 1{,}11 + n\cdot\pi\\ \vee\\ x = -\frac{\pi}{4} + n\cdot\pi\end{cases} \Leftrightarrow \begin{cases}\underline{\underline{x\approx 1{,}11}} \quad \vee \quad \underline{\underline{x\approx 4{,}25}}\\ \vee\\ \underline{\underline{x = \frac{3\pi}{4}}} \quad \vee \quad \underline{\underline{x = \frac{7\pi}{4}}}\end{cases}$$

b) $5\sin^2 x + 4\sin x\cdot\cos x + \cos^2 x = 1 \qquad x\in[0,2\pi]$

$\Leftrightarrow\ 5\sin^2 x + 4\sin x\cdot\cos x + \cos^2 x = 1\cdot\left(\sin^2 x + \cos^2 x\right)$

$\Leftrightarrow\ 5\sin^2 x + 4\sin x\cdot\cos x + \cos^2 x = \sin^2 x + \cos^2 x$

$\Leftrightarrow\ 5\sin^2 x + 4\sin x\cdot\cos x + \cos^2 x - \sin^2 x - \cos^2 x = 0$

$\Leftrightarrow\ 4\sin^2 x + 4\sin x\cdot\cos x = 0 \quad \overset{\cos^2 x\neq 0}{\Leftrightarrow} \quad \frac{4\sin^2 x}{\cos^2 x} + 4\frac{\sin x\cdot\cos x}{\cos^2 x} = \frac{0}{\cos^2 x}$

$\Leftrightarrow\ 4\tan^2 x + 4\tan x = 0 \quad \Leftrightarrow\ 4\tan x\cdot(\tan x + 1) = 0 \quad \Leftrightarrow \quad 4\tan x = 0 \ \vee\ \tan x + 1 = 0$

$$\Leftrightarrow \begin{cases}\tan x = 0\\ \vee\\ \tan x = -1\end{cases} \Leftrightarrow \begin{cases}x = \tan^{-1} 0\\ \vee\\ x = \tan^{-1}(-1)\end{cases} \Leftrightarrow \begin{cases}x = 0 + n\cdot\pi\\ \vee\\ x = -\frac{\pi}{4} + n\cdot\pi\end{cases} \Leftrightarrow \begin{cases}\underline{\underline{x = 0}} \quad \vee \quad \underline{\underline{x = \pi}}\\ \vee\\ \underline{\underline{x = \frac{3\pi}{4}}} \quad \vee \quad \underline{\underline{x = \frac{7\pi}{4}}}\end{cases}$$

c) $4\sin^2 x - 4\sqrt{3}\sin x \cdot \cos x + 4\cos^2 x = 1 \qquad x \in [-\pi, \pi]$

$$\Leftrightarrow\ 4\sin^2 x - 4\sqrt{3}\sin x \cdot \cos x + 4\cos^2 x = 1 \cdot \left(\sin^2 x + \cos^2 x\right)$$

$$\Leftrightarrow\ 4\sin^2 x - 4\sqrt{3}\sin x \cdot \cos x + 4\cos^2 x = \sin^2 x + \cos^2 x$$

$$\Leftrightarrow\ 4\sin^2 x - 4\sqrt{3}\sin x \cdot \cos x + 4\cos^2 x - \sin^2 x - \cos^2 x = 0$$

$$\Leftrightarrow\ 3\sin^2 x - 4\sqrt{3}\sin x \cdot \cos x + 3\cos^2 x = 0 \ \overset{\cos^2 x \neq 0}{\Leftrightarrow}\ \frac{3\sin^2 x}{\cos^2 x} - \frac{4\sqrt{3}\sin x \cdot \cos x}{\cos^2 x} + \frac{3\cos^2 x}{\cos^2 x} = \frac{0}{\cos^2 x}$$

$$\Leftrightarrow\ 3\tan^2 x - 4\sqrt{3}\tan x + 3 = 0 \quad \Leftrightarrow\ \tan x = \frac{-\left(-4\sqrt{3}\right) \pm \sqrt{\left(-4\sqrt{3}\right)^2 - 4\cdot 3\cdot 3}}{2\cdot 3} = \frac{4\sqrt{3} \pm \sqrt{12}}{6}$$

$$\Leftrightarrow\ \tan x = \frac{4\sqrt{3} \pm 2\cdot\sqrt{3}}{6}$$

$$\Leftrightarrow \begin{cases} \tan x = \sqrt{3} \\ \vee \\ \tan x = \frac{\sqrt{3}}{3} \end{cases} \Leftrightarrow \begin{cases} x = \tan^{-1}\sqrt{3} \\ \vee \\ x = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) \end{cases} \Leftrightarrow \begin{cases} x = \frac{\pi}{3} + n\cdot\pi \\ \vee \\ x = \frac{\pi}{6} + n\cdot\pi \end{cases} \Leftrightarrow \begin{cases} \underline{\underline{x = -\frac{2\pi}{3}}} \ \vee \ \underline{\underline{x = \frac{\pi}{3}}} \\ \vee \\ \underline{\underline{x = -\frac{5\pi}{6}}} \ \vee \ \underline{\underline{x = \frac{\pi}{6}}} \end{cases}$$

3.1 Sinusfunksjonen

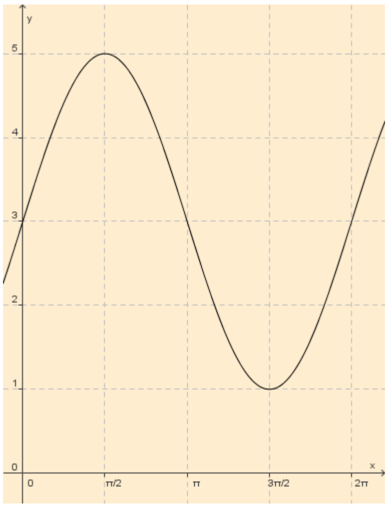
Oppgave 3.10

a) $f(x) = 3 + 2\sin x \quad x \in [0, 2\pi]$

$f(x)_{maks} = 3 + 2 \cdot 1 = \underline{\underline{5}}$ når $\sin x = 1 \Leftrightarrow \underline{\underline{x = \frac{\pi}{2}}}$

b) $f(x)_{\min} = 3 + 2 \cdot (-1) = \underline{\underline{1}}$ når $\sin x = -1 \Leftrightarrow \underline{\underline{x = \frac{3\pi}{2}}}$

c)



Oppgave 3.11

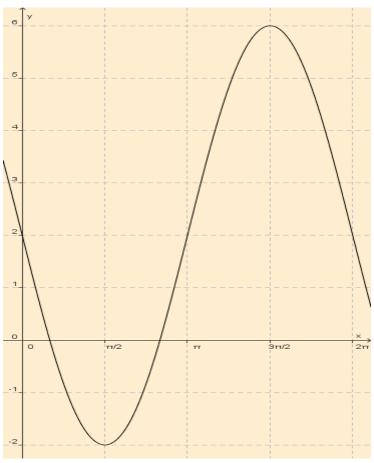
a) $f(x) = 2 - 4\sin x \quad x \in [0, 2\pi]$

Nullpunkter der $f(x) = 0 \Rightarrow 2 - 4\sin x = 0 \Leftrightarrow \sin x = \frac{1}{2} \Leftrightarrow \underline{\underline{x = \frac{\pi}{6}}} \vee x = \pi - \frac{\pi}{6} = \underline{\underline{\frac{5\pi}{6}}}$

b) $f(x)_{maks} = 2 - 4 \cdot (-1) = \underline{\underline{6}}$ når $\sin x = -1 \Leftrightarrow \underline{\underline{x = \frac{3\pi}{2}}}$

c) $f(x)_{\min} = 2 - 4 \cdot 1 = \underline{\underline{-2}}$ når $\sin x = 1 \Leftrightarrow \underline{\underline{x = \frac{\pi}{2}}}$

d)



Oppgave 3.12

a) $f(x)=1+2\sin\pi x \quad x\in[0,2]$

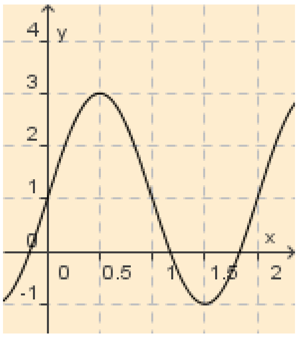
Nullpunkter der $f(x)=0 \;\Rightarrow\; 1+2\sin\pi x=0 \;\Leftrightarrow\; \sin\pi x=-\frac{1}{2} \;\Leftrightarrow$

$\pi x=2\pi-\frac{\pi}{6} \;\vee\; \pi x=\pi+\frac{\pi}{6} \;\Leftrightarrow\; \pi x=\frac{11\pi}{6} \;\vee\; \pi x=\frac{7\pi}{6} \;\overset{:\pi}{\Leftrightarrow}\; \underline{\underline{x=\frac{11}{6}}} \;\vee\; \underline{\underline{x=\frac{7}{6}}}$

b) $f(x)_{maks}=1+2\cdot 1=\underline{\underline{3}}$ når $\sin\pi x=1 \;\Leftrightarrow\; \pi x=\frac{\pi}{2} \;\Leftrightarrow\; \underline{\underline{x=\frac{1}{2}}}$

c) $f(x)_{\min}=1+2\cdot(-1)=\underline{\underline{-1}}$ når $\sin\pi x=-1 \;\Leftrightarrow\; \pi x=\frac{3\pi}{2} \;\Leftrightarrow\; \underline{\underline{x=\frac{3}{2}}}$

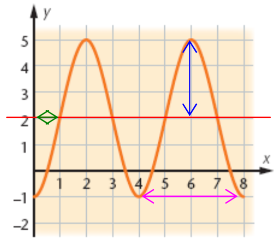
d)



3.2 Amplitude, periode og likevektslinje

Oppgave 3.21

a)

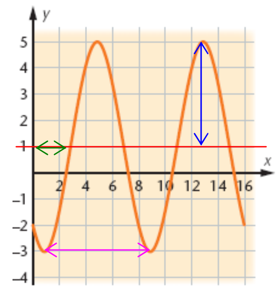


Likevektslinje: $y = 2$ Amplitude: $A = 3$

Periode: $p = 4 \Rightarrow k = \frac{2\pi}{4} = \frac{\pi}{2}$ Faseforskyvning: $c = 1$

Funksjonsuttrykket: $\underline{\underline{f(x) = 3\sin\left(\frac{\pi}{2}(x-1)\right) + 2}}$

b)

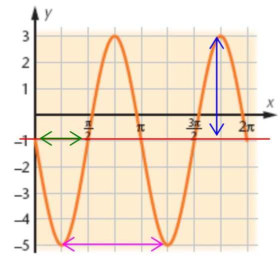


Likevektslinje: $y = 1$ Amplitude: $A = 4$

Periode: $p = 8 \Rightarrow k = \frac{2\pi}{8} = \frac{\pi}{4}$ Faseforskyvning: $c = 3$

Funksjonsuttrykket: $\underline{\underline{f(x) = 4\sin\left(\frac{\pi}{4}(x-3)\right) + 1}}$

c)



Likevektslinje: $y = -1$ Amplitude: $A = 4$

Periode: $p = \pi \Rightarrow k = \frac{2\pi}{\pi} = 2$ Faseforskyvning: $c = \frac{\pi}{2}$

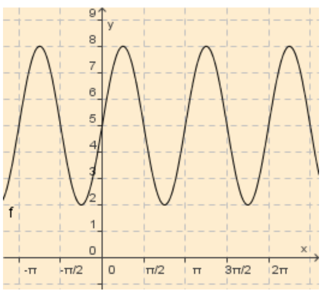
Funksjonsuttrykket: $\underline{\underline{f(x) = 4\sin\left(2\left(x - \frac{\pi}{2}\right)\right) - 1}}$

eller $\underline{\underline{f(x) = 4\sin(2x - \pi) - 1}}$

eller $\underline{\underline{f(x) = -4\sin 2x - 1}}$ $[\sin(2x - \pi) = -\sin 2x]$

Oppgave 3.22

a) $f(x) = 3\sin 2x + 5$

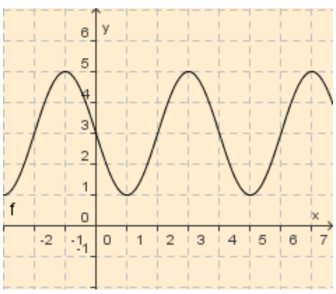


Amplitude: $\underline{\underline{A = 3}}$

Likevektslinje: $\underline{\underline{y = 5}}$

Periode: $p = \frac{2\pi}{2} = \underline{\underline{\pi}}$

b) $f(x) = -2\sin\left(\frac{\pi}{2}x\right) + 3$

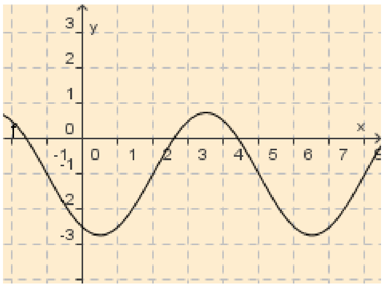


Amplitude: $\underline{\underline{A = |-2| = 2}}$

Likevektslinje: $\underline{\underline{y = 3}}$

Periode: $p = \frac{2\pi}{\frac{\pi}{2}} = \underline{\underline{4}}$

c) $f(x)=\sqrt{3}\sin\left(\frac{\pi}{3}(x-2)\right)-1$



Amplitude: $\underline{\underline{A=\sqrt{3}}}$

Likevektslinje: $\underline{\underline{y=-1}}$

Periode: $p=\frac{2\pi}{\frac{\pi}{3}}=\underline{\underline{6}}$

Oppgave 3.23

a) $f(x)=3+6\sin(\pi x) \qquad x\in[0,4]$

Amplitude: $\underline{\underline{A=6}}$ Likevektslinje: $\underline{\underline{y=3}}$ Periode: $p=\frac{2\pi}{\pi}=\underline{\underline{2}}$

b) $f(x)_{maks}=3+6\cdot 1=\underline{\underline{9}}$ når $\sin(\pi x)=1 \Leftrightarrow$

$\pi x=\frac{\pi}{2}+n\cdot 2\pi \Leftrightarrow x=\frac{1}{2}+2n \Leftrightarrow \underline{\underline{x=\frac{1}{2}}} \vee \underline{\underline{x=\frac{5}{2}}}$

c) $f(x)_{\min}=3+6\cdot(-1)=\underline{\underline{-3}}$ når $\sin(\pi x)=-1 \Leftrightarrow$

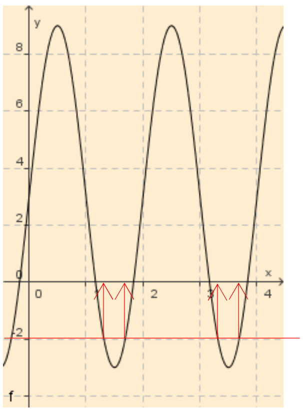
$\pi x=\frac{3\pi}{2}+n\cdot 2\pi \Leftrightarrow x=\frac{3}{2}+2n \Leftrightarrow \underline{\underline{x=\frac{3}{2}}} \vee \underline{\underline{x=\frac{7}{2}}}$

d) $3+6\sin(\pi x)=0 \Leftrightarrow \sin(\pi x)=-\frac{1}{2} \Leftrightarrow \pi x=-\frac{\pi}{6}+n\cdot 2\pi \vee \pi x=\left(\pi+\frac{\pi}{6}\right)+n\cdot 2\pi \Leftrightarrow$

$\pi x=-\frac{\pi}{6}+n\cdot 2\pi \vee \pi x=\frac{7\pi}{6}+n\cdot 2\pi \Leftrightarrow x=-\frac{1}{6}+2n \vee x=\frac{7}{6}+2n$

Nullpunkter: $\underline{\underline{x=\frac{7}{6}}}$, $\underline{\underline{x=\frac{11}{6}}}$, $\underline{\underline{x=\frac{19}{6}}}$ og $\underline{\underline{x=\frac{23}{6}}}$

e)



f) $f(x)=-2$ løst grafisk: $\underline{\underline{x\approx 1{,}3}}$, $\underline{\underline{x\approx 1{,}7}}$, $\underline{\underline{x\approx 3{,}3}}$ og $\underline{\underline{x\approx 3{,}7}}$

Løst ved regning:

$3+6\sin(\pi x)=-2 \quad\Leftrightarrow\quad \sin(\pi x)=-\frac{5}{6} \quad\Leftrightarrow$

$\pi x \approx -0{,}99+n\cdot 2\pi \quad\vee\quad \pi x \approx \pi+0{,}99+n\cdot 2\pi \quad\Leftrightarrow$

$x \approx -0{,}32+2n \quad\vee\quad x\approx 1{,}32+2n \quad\Leftrightarrow$

$\underline{\underline{x\approx 1{,}32}} \quad\vee\quad \underline{\underline{x\approx 1{,}68}} \quad\vee\quad \underline{\underline{x\approx 3{,}32}} \quad\vee\quad \underline{\underline{x\approx 3{,}68}}$

Oppgave 3.24

a) $T(x)=12+6\sin\left(\frac{2\pi}{365}(x-82)\right) \qquad x\in[1,365]$

Amplitude: $\underline{\underline{A=6}}$ Likevektslinje: $\underline{\underline{y=12}}$ Periode: $p=\dfrac{2\pi}{\frac{2\pi}{365}}=\underline{\underline{365}}$

Tallene forteller at gjennomsnittlig antall timer med dagslys er 12.
Minimum er 6 timer og maksimum 18 timer dagslys i døgnet over en periode på ett år.

b) $T(1)=12+6\sin\left(\frac{2\pi}{365}(1-82)\right)\approx 6{,}09\text{timer}\approx 6\text{t}+\underbrace{0{,}09\cdot 60\text{min}}_{\approx 5}$

<u>1.januar er det dagslys i 6 timer og 5 minutter.</u>

$T(182)=12+6\sin\left(\frac{2\pi}{365}(182-82)\right)\approx 17{,}93\text{timer}\approx 17\text{t}+\underbrace{0{,}93\cdot 60\text{min}}_{\approx 55}$

<u>1.juli er det dagslys i 17 timer og 55 minutter.</u>

c) Antall timer dagslys den mørkeste dagen: $12+6\cdot(-1)=\underline{\underline{6}}$
Antall timer dagslys den lyseste dagen: $12+6\cdot 1=\underline{\underline{18}}$

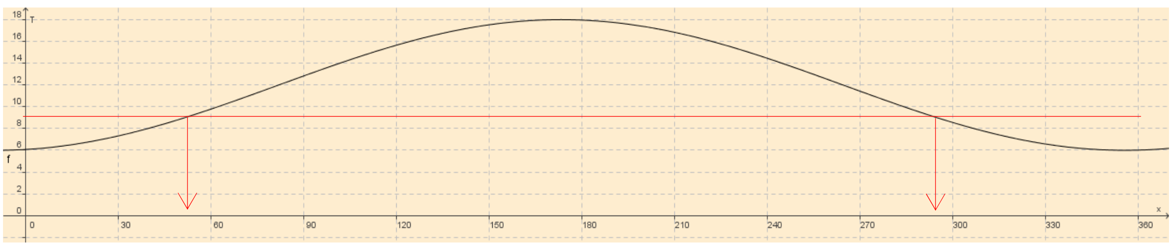
d) Mørkest når $\sin\left(\frac{2\pi}{365}(x-82)\right)=-1 \quad\Leftrightarrow\quad \frac{2\pi}{365}(x-82)=\frac{3\pi}{2}+n\cdot 2\pi \quad\overset{:\frac{2\pi}{365}}{\Leftrightarrow}$

$x-82\approx 274+365n \quad\Leftrightarrow\quad x=356+365n \quad\Leftrightarrow\quad x=356$ <u>Mørkeste dag er nr. 356 i året (22.des).</u>

Lysest når $\sin\left(\frac{2\pi}{365}(x-82)\right)=1 \quad\Leftrightarrow\quad \frac{2\pi}{365}(x-82)=\frac{\pi}{2}+n\cdot 2\pi \quad\overset{:\frac{2\pi}{365}}{\Leftrightarrow}$

$x-82\approx 91+365n \quad\Leftrightarrow\quad x=173+365n \quad\Leftrightarrow\quad x=173$ <u>Lyseste dag er nr. 173 i året (21.juni).</u>

e)



f) Grafisk løst: 9 timer dagslys på dag nr. 52 (20.feb) og dag nr. 295 (21.okt).

$$12+6\sin\left(\tfrac{2\pi}{365}\left(x-82\right)\right)=9 \quad \Leftrightarrow \quad \sin\left(\tfrac{2\pi}{365}\left(x-82\right)\right)=-\tfrac{1}{2} \quad \Leftrightarrow$$

$$\tfrac{2\pi}{365}\left(x-82\right)=-\tfrac{\pi}{6}+n\cdot 2\pi \quad \vee \quad \tfrac{2\pi}{365}\left(x-82\right)=\pi+\tfrac{\pi}{6}+n\cdot 2\pi \quad \overset{:\frac{2\pi}{365}}{\Leftrightarrow}$$

$$x-82\approx -30+365n \quad \vee \quad x-82\approx 213+365n \quad \Leftrightarrow$$

$$x=52+365n \quad \vee \quad x\approx 295+365n \Leftrightarrow \quad x=52 \quad \vee \quad x=295$$

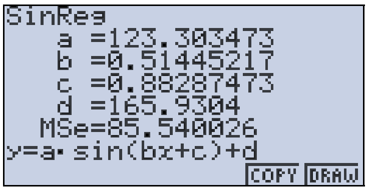
9 timer dagslys på dag nr.52 og dag nr. 295.

3.3 Trigonometriske modeller

Oppgave 3.30

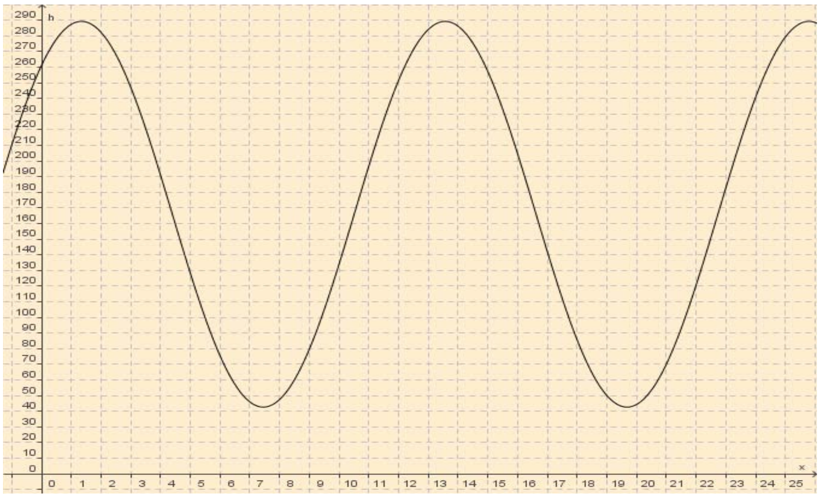
a)

x	0	2	4	6	8	10	12	14	16	18	20	22	24
h (cm)	270	295	182	72	48	121	239	290	198	85	58	127	241



$$\underline{\underline{h(x) = 123,3\sin(0,514x + 0,882) + 165,9}}$$

b)



c) $h(17) = 123,3\sin(0,514 \cdot 17 + 0,882) + 165,9 = 141,98 \approx 142$

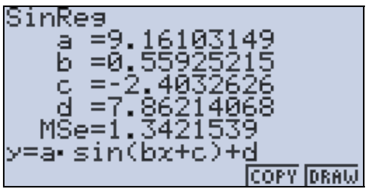
Siden målt vannstand var 142 cm over nullnivået, stemmer modellen bra.

d) Normal vannstand i Trondheim er 165,9cm over nullnivået.
Vannstanden varierer i intervallet $[165,9 - 123,3 \;\; , \;\; 165,9 + 123,3] = [42,6 \;\; , \;\; 289,2]$
med en periode på $\frac{2\pi}{0,514} \approx 12$ timer

Oppgave 3.31

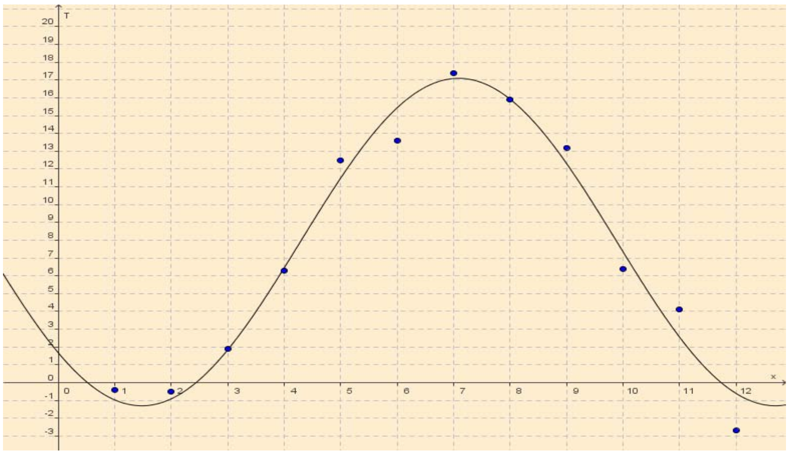
a)

x	1	2	3	4	5	6	7	8	9	10	11	12
T (°C)	–0,4	–0,5	1,9	6,3	12,5	13,6	17,4	15,9	13,2	6,4	4,1	–2,7



$$\underline{\underline{T(x) = 9,2\sin(0,56x - 2,4) + 7,9}}$$

b) + **c)**



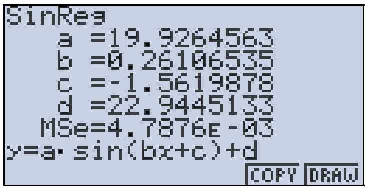
Modellen stemmer relativt bra med de målte dataene.

d) Middeltemperaturen på Blindern er $7,9°C$ og variasjonen i løpet av et år er $\pm 9,2°C$.

Oppgave 3.32

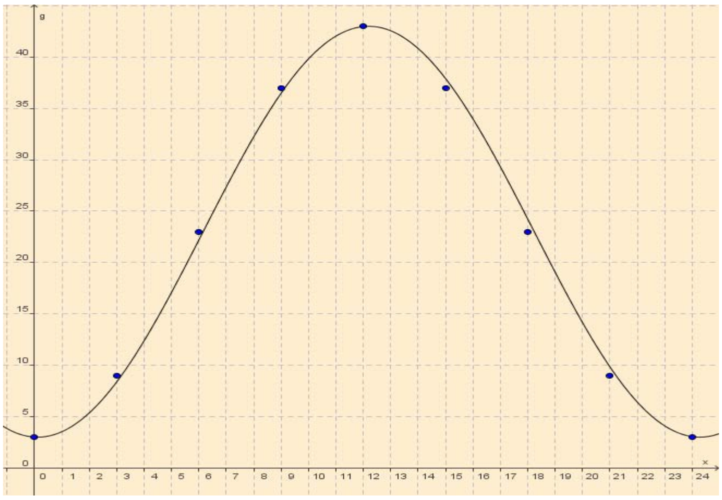
a)

x	0	3	6	9	12	15	18	21	24
$g(°)$	3	9	23	37	43	37	23	9	3



$$g(x) = 20\sin(0,26x - 1,6) + 23$$

b) + **c)**



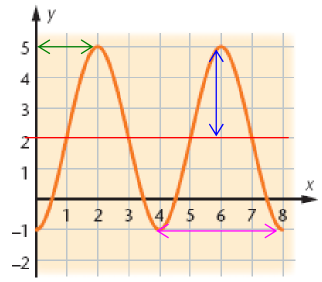
Modellen stemmer bra med de målte dataene.

d) Gjennomsnittlig solhøyde dette døgnet var 23^{o} over horisonten.

3.4 Cosinusfunksjonen

Oppgave 3.41

a)

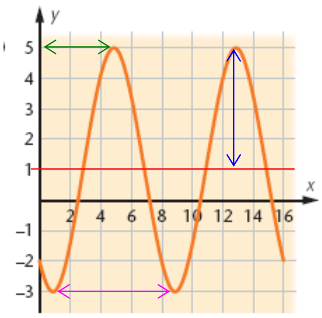


Likevektslinje: $y = 2$ Amplitude: $A = 3$

Periode: $p = 4 \Rightarrow k = \frac{2\pi}{4} = \frac{\pi}{2}$ Faseforskyvning: $c = 2$

Funksjonsuttrykket: $\underline{\underline{f(x) = 3\cos\left(\frac{\pi}{2}(x-2)\right) + 2}}$

b)

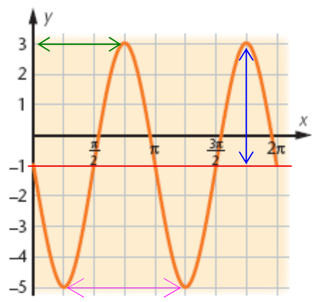


Likevektslinje: $y = 1$ Amplitude: $A = 4$

Periode: $p = 8 \Rightarrow k = \frac{2\pi}{8} = \frac{\pi}{4}$ Faseforskyvning: $c = 5$

Funksjonsuttrykket: $\underline{\underline{f(x) = 4\cos\left(\frac{\pi}{4}(x-5)\right) + 1}}$

Oppgave 3.42



Likevektslinje: $y = -1$ Amplitude: $A = 4$

Periode: $p = \pi \Rightarrow k = \frac{2\pi}{\pi} = 2$ Faseforskyvning: $c = \frac{3\pi}{4}$

Funksjonsuttrykket: $\underline{\underline{f(x) = 4\cos\left(2\left(x - \frac{3\pi}{4}\right)\right) - 1}}$

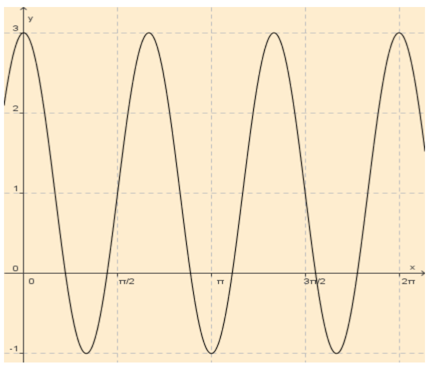
Oppgave 3.43

a) $f(x) = 2\cos 3x + 1$

Amplitude: $\underline{\underline{A = 2}}$

Likevektslinje: $\underline{\underline{y = 1}}$

Periode: $\underline{\underline{p = \frac{2\pi}{3}}}$

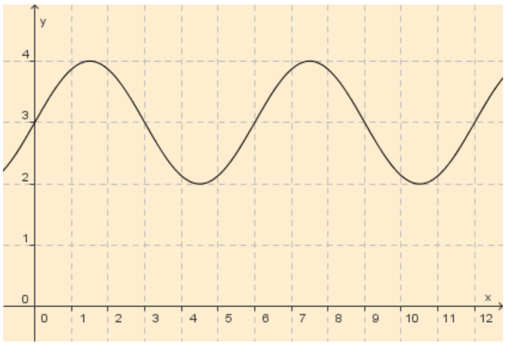


b) $f(x)=3-\cos\left(\frac{\pi}{3}x+\frac{\pi}{2}\right)=3-\cos\left(\frac{\pi}{3}\left(x+\frac{3}{2}\right)\right)$

Amplitude: $\underline{\underline{A=|-1|=1}}$

Likevektslinje: $\underline{\underline{y=3}}$

Periode: $p=\dfrac{2\pi}{\frac{\pi}{3}}=\underline{\underline{6}}$

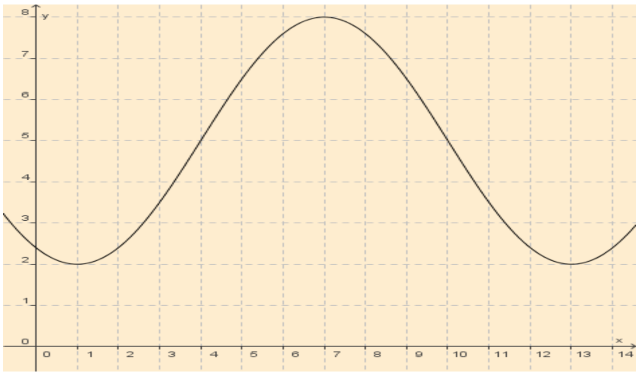


c) $f(x)=5-3\cos\left(\frac{\pi}{6}(x-1)\right)$

Amplitude: $\underline{\underline{A=|-3|=3}}$

Likevektslinje: $\underline{\underline{y=5}}$

Periode: $p=\dfrac{2\pi}{\frac{\pi}{6}}=\underline{\underline{12}}$



Oppgave 3.44

a) $f(x)=-1+2\cos\left(\frac{\pi}{6}x\right)\quad x\in[0,12]$

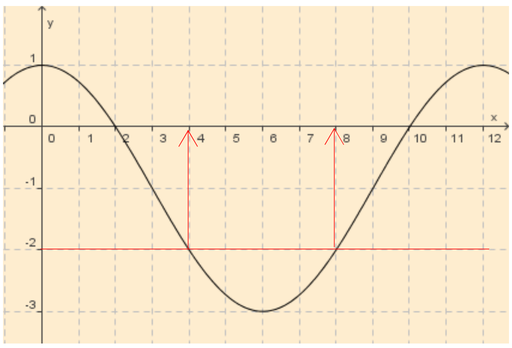
$-1+2\cos\left(\frac{\pi}{6}x\right)=0 \Leftrightarrow \cos\left(\frac{\pi}{6}x\right)=\frac{1}{2} \Leftrightarrow \frac{\pi}{6}x=\frac{\pi}{3}+n\cdot 2\pi \vee \frac{\pi}{6}x=-\frac{\pi}{3}+n\cdot 2\pi \overset{:\frac{\pi}{6}}{\Leftrightarrow}$

$x=2+12n \vee x=-2+12n$ Nullpunkter: $\underline{\underline{x=2}}$ og $\underline{\underline{x=10}}$

b) $f(x)_{maks}=-1+2\cdot 1=\underline{\underline{1}}$ når $\cos\left(\frac{\pi}{6}x\right)=1 \Leftrightarrow \frac{\pi}{6}x=0+n\cdot 2\pi \Leftrightarrow x=12n \Leftrightarrow \underline{\underline{x=0}} \vee \underline{\underline{x=12}}$

c) $f(x)_{\min}=-1+2\cdot(-1)=\underline{\underline{-3}}$ når $\cos\left(\frac{\pi}{6}x\right)=-1 \Leftrightarrow \frac{\pi}{6}x=\pi+n\cdot 2\pi \Leftrightarrow x=6+12n \Leftrightarrow \underline{\underline{x=6}}$

d)



e) $f(x)=-2$ løst grafisk: $\underline{\underline{x=4}}$ og $\underline{\underline{x=8}}$

Løst ved regning:

$-1+2\cos\left(\frac{\pi}{6}x\right)=-2 \Leftrightarrow \cos\left(\frac{\pi}{6}x\right)=-\frac{1}{2} \Leftrightarrow$

$\frac{\pi}{6}x=\frac{2\pi}{3}+n\cdot 2\pi \vee \frac{\pi}{6}x=-\frac{2\pi}{3}+n\cdot 2\pi \overset{:\frac{\pi}{6}}{\Leftrightarrow}$

$x=4+12n \vee x=-4+12n \Leftrightarrow$

$\underline{\underline{x=4}} \vee \underline{\underline{x=8}}$

Oppgave 3.45

a) $h(x) = 23 - 20\cos\left(\frac{\pi}{12}x\right)$

$h(6) = 23 - 20\cos\left(\frac{\pi}{12}\cdot 6\right) = 23$ Solhøyden kl.6.00 var 23°.

$h(14) = 23 - 20\cos\left(\frac{\pi}{12}\cdot 14\right) \approx 40{,}3°$ Solhøyden kl.14.00 var 40,3°.

$h(22) = 23 - 20\cos\left(\frac{\pi}{12}\cdot 22\right) \approx 5{,}7°$ Solhøyden kl.22.00 var 5,7°.

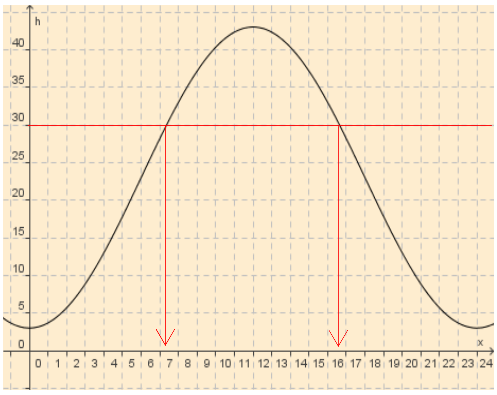
b) $h(x)_{maks} = 23 - 20\cdot(-1) = 43$ når $\cos\left(\frac{\pi}{12}x\right) = -1 \Leftrightarrow \frac{\pi}{12}x = \pi + n\cdot 2\pi$

$\Leftrightarrow x = 12 + 24n \Leftrightarrow x = 12$ Sola står maksimale 43° over horisonten kl.12.

c) $h(x)_{\min} = 23 - 20\cdot 1 = 3$ når $\cos\left(\frac{\pi}{12}x\right) = 1 \Leftrightarrow \frac{\pi}{12}x = 0 + n\cdot 2\pi$

$\Leftrightarrow x = 24n \Leftrightarrow x = 24$ Sola står minimale 3° over horisonten kl.24.

d)



e) $h(x) = 30$ løst grafisk: $x \approx 7{,}4$ og $x \approx 16{,}7$

Løst ved regning:

$23 - 20\cos\left(\frac{\pi}{12}x\right) = 30 \Leftrightarrow \cos\left(\frac{\pi}{12}x\right) = -\frac{7}{20} \Leftrightarrow$

$\frac{\pi}{12}x \approx 1{,}93 + n\cdot 2\pi \quad \vee \quad \frac{\pi}{12}x \approx -1{,}93 + n\cdot 2\pi \quad \Leftrightarrow^{:\frac{\pi}{12}}$

$x \approx 7{,}37 + 24n \quad \vee \quad x \approx -7{,}37 + 24n \quad \Leftrightarrow$

$x \approx 7{,}37 = 7t + 0{,}37\cdot 60\,\text{min} \approx 7.22 \quad \vee$

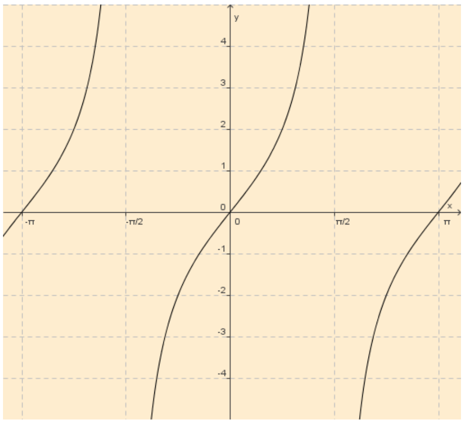
$x \approx 16{,}63 = 16t + 0{,}63\cdot 60\,\text{min} \approx 16.38$

Sola sto 30° over horisonten kl.7.22 og kl.16.38.

3.5 Tangensfunksjonen

Oppgave 3.50

a) $f(x) = 2\tan x \quad x \in [-\pi, \pi]$



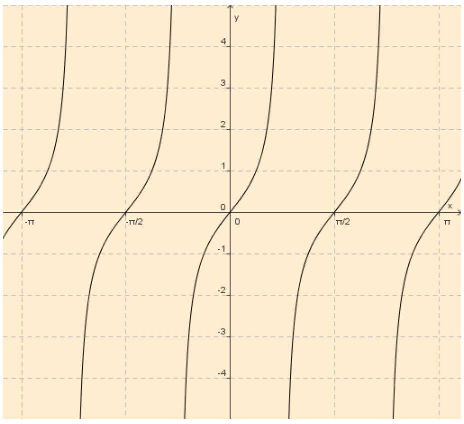
$2\tan x = 0 \quad \Leftrightarrow \quad \tan x = 0 \quad \Leftrightarrow \quad x = 0 + n \cdot \pi$

$\Rightarrow$ Nullpunkter: $\underline{\underline{x = -\pi}}$, $\underline{\underline{x = 0}}$ og $\underline{\underline{x = \pi}}$

Tan ikke definert der $x = \frac{\pi}{2} + n \cdot \pi$

$\Rightarrow$ Bruddpunkter: $\underline{\underline{x = -\frac{\pi}{2}}}$ og $\underline{\underline{x = \frac{\pi}{2}}}$

b) $f(x) = \tan 2x \quad x \in [-\pi, \pi]$



$\tan 2x = 0 \quad \Leftrightarrow \quad 2x = 0 + n \cdot \pi \quad \Leftrightarrow \quad x = n \cdot \frac{\pi}{2}$

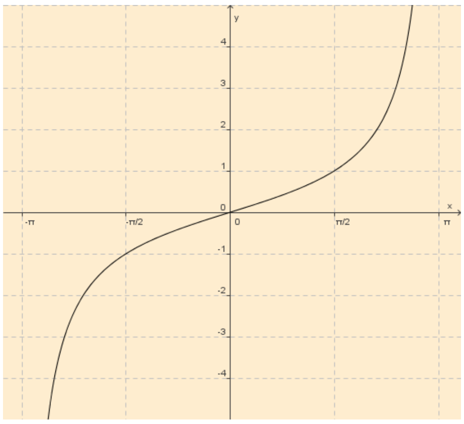
$\Rightarrow$ Nullpunkter: $\underline{\underline{x = \pm\pi}}$, $\underline{\underline{x = \pm\frac{\pi}{2}}}$ og $\underline{\underline{x = 0}}$

Tan ikke definert der $2x = \frac{\pi}{2} + n \cdot \pi \quad \Leftrightarrow$

$x = \frac{\pi}{4} + n \cdot \frac{\pi}{2} \quad \Rightarrow$

Bruddpunkter: $\underline{\underline{x = \pm\frac{3\pi}{4}}}$ og $\underline{\underline{x = \pm\frac{\pi}{4}}}$

c) $f(x) = \tan \frac{x}{2} \quad x \in [-\pi, \pi]$



$\tan \frac{x}{2} = 0 \quad \Leftrightarrow \quad \frac{x}{2} = 0 + n \cdot \pi \quad \Leftrightarrow \quad x = 2n \cdot \pi$

$\Rightarrow$ Nullpunkt: $\underline{\underline{x = 0}}$

Tan ikke definert der $\frac{x}{2} = \frac{\pi}{2} + n \cdot \pi \quad \Leftrightarrow$

$x = \pi + 2n \cdot \pi \quad \Rightarrow$

Bruddpunkter: $\underline{\underline{x = \pm\pi}}$

Oppgave 3.51

a) $f(x)=4-4\tan\left(\frac{\pi}{2}x\right) \quad x\in[-1,3]$

$4-4\tan\left(\frac{\pi}{2}x\right)=0 \Leftrightarrow \tan\left(\frac{\pi}{2}x\right)=1 \Leftrightarrow \frac{\pi}{2}x=\frac{\pi}{4}+n\cdot\pi \overset{:\frac{\pi}{2}}{\Leftrightarrow} x=\frac{1}{2}+2n$

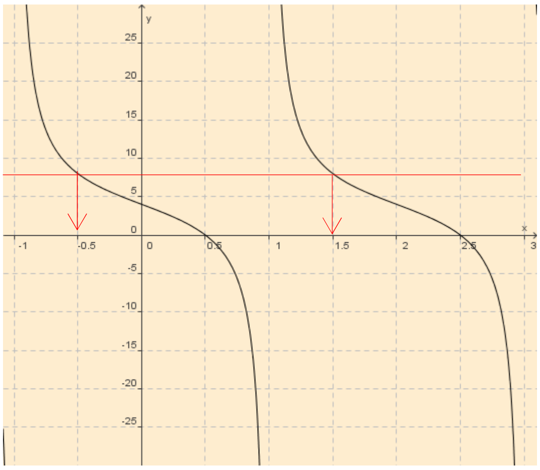
$\Rightarrow$ Nullpunkter: $\underline{\underline{x=\frac{1}{2}}}$ og $\underline{\underline{x=\frac{5}{2}}}$

b) Asymptoter der $\frac{\pi}{2}x=\frac{\pi}{2}+n\cdot\pi \Leftrightarrow$

$x=1+2n \Rightarrow$

Asymptoter: $\underline{\underline{x=-1}}$, $\underline{\underline{x=1}}$ og $\underline{\underline{x=3}}$

c)



d) $f(x)=8$ løst grafisk: $\underline{\underline{x=-\frac{1}{2}}}$ og $\underline{\underline{x=\frac{3}{2}}}$

Løst ved regning:

$4-4\tan\left(\frac{\pi}{2}x\right)=8 \Leftrightarrow \tan\left(\frac{\pi}{2}x\right)=-1 \Leftrightarrow$

$\frac{\pi}{2}x=-\frac{\pi}{4}+n\cdot\pi \overset{:\frac{\pi}{2}}{\Leftrightarrow} x=-\frac{1}{2}+2n$

$\underline{\underline{x=-\frac{1}{2}}} \vee \underline{\underline{x=\frac{3}{2}}}$

3.6 Derivasjon av de trigonometriske funksjonene

Oppgave 3.60

a) $f(x) = 2\sin x + 1 \quad \Rightarrow \quad \underline{\underline{f'(x) = 2\cos x}}$

b) $f(t) = 2t + 3\sin t \quad \Rightarrow \quad \underline{\underline{f'(t) = 2 + 3\cos t}}$

c) $f(x) = x\sin x \quad \Rightarrow \quad f'(x) = 1\cdot\sin x + x\cdot\cos x = \underline{\underline{\sin x + x\cos x}}$

d) $f(t) = \dfrac{2}{\sin t} \quad \Rightarrow \quad f'(t) = \dfrac{0\cdot\sin t - 2\cdot\cos t}{\sin^2 t} = \underline{\underline{-\dfrac{2\cos t}{\sin^2 t}}}$

Oppgave 3.61

a) $g(t) = 1 - 2\cos t \quad \Rightarrow \quad g'(t) = -2\cdot(-\sin t) = \underline{\underline{2\sin t}}$

b) $g(t) = 2\sin t + 3\cos t \quad \Rightarrow \quad g'(t) = 2\cdot\cos t + 3\cdot(-\sin t) = \underline{\underline{2\cos t - 3\sin t}}$

c) $g(t) = 5\tan t - 5t \quad \Rightarrow \quad g'(t) = 5\cdot(1 + \tan^2 t) - 5 = 5 + 5\tan^2 t - 5 = \underline{\underline{5\tan^2 t}}$

d) $g(t) = \dfrac{\cos t}{\sin t} \quad \Rightarrow \quad g'(t) = \dfrac{-\sin t\cdot\sin t - \cos t\cdot\cos t}{\sin^2 t} = \dfrac{-\sin^2 t - \cos^2 t}{\sin^2 t}$

$$= \frac{-1\cdot(\sin^2 t + \cos^2 t)}{\sin^2 t} = \underline{\underline{-\frac{1}{\sin^2 t}}}$$

Oppgave 3.62

a) $f(x) = \sin 2x \quad \overset{u(x)=2x}{\Rightarrow} \quad f'(x) = \cos 2x\cdot 2 = \underline{\underline{2\cos 2x}}$

b) $f(x) = \sin x^2 \quad \overset{u(x)=x^2}{\Rightarrow} \quad f'(x) = \cos x^2\cdot 2x = \underline{\underline{2x\cos x^2}}$

c) $f(x) = \sin^2 x = (\sin x)^2 \quad \overset{u(x)=\sin x}{\Rightarrow} \quad f'(x) = 2\sin x\cdot\cos x = \underline{\underline{2\sin x\cos x}}$

Oppgave 3.63

a) $f(x) = 2\cos \pi x \quad \overset{u(x)=\pi x}{\Rightarrow} \quad f'(x) = 2\cdot(-\sin \pi x)\cdot\pi = \underline{\underline{-2\pi\sin \pi x}}$

b) $f(x) = 9\sin \frac{\pi}{3}x \quad \overset{u(x)=\frac{\pi}{3}x}{\Rightarrow} \quad f'(x) = 9\cdot\cos\frac{\pi}{3}x\cdot\frac{\pi}{3} = \underline{\underline{3\pi\cos\frac{\pi}{3}x}}$

c)

$$f(x)=x\cos\tfrac{\pi}{6}x \overset{u(x)=\frac{\pi}{6}x}{\Rightarrow} f'(x)=1\cdot\cos\tfrac{\pi}{6}x+x\cdot\left(-\sin\tfrac{\pi}{6}x\right)\cdot\tfrac{\pi}{6}=\underline{\underline{\cos\tfrac{\pi}{6}x-\tfrac{\pi}{6}x\sin\tfrac{\pi}{6}x}}$$

d)

$$f(x)=x^2\cos(\pi x+5) \overset{u(x)=\pi x+5}{\Rightarrow} f'(x)=2x\cdot\cos(\pi x+5)+x^2\cdot\left(-\sin(\pi x+5)\right)\cdot\pi$$

$$=\underline{\underline{2x\cos(\pi x+5)-\pi x^2\sin(\pi x+5)}}$$

Oppgave 3.64

a)

$$N(x)=200-3x-40\cos\left(\tfrac{\pi}{6}x\right) \qquad x\in[0,24]$$

$N(0)=200-3\cdot 0-40\cos\left(\tfrac{\pi}{6}\cdot 0\right)=200-40=160$ <u>Medlemstallet 1.januar 2008 vil være 160.</u>

$N(6)=200-3\cdot 6-40\cos\left(\tfrac{\pi}{6}\cdot 6\right)=200-18-40\cdot(-1)=222$ <u>Medlemstallet 1.juli 2008 vil være 222.</u>

$N(12)=200-3\cdot 12-40\cos\left(\tfrac{\pi}{6}\cdot 12\right)=200-36-40\cdot 1=124$ <u>Medlemstallet 1.januar 2009 vil være 124.</u>

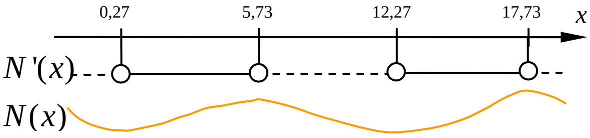
b)

$$N'(x)=-3-40\cdot\left(-\sin\left(\tfrac{\pi}{6}x\right)\right)\cdot\tfrac{\pi}{6}=-3+\tfrac{20\pi}{3}\sin\left(\tfrac{\pi}{6}x\right)$$

Topp- og bunnpunkter der $N'(x)=0 \;\Rightarrow$

$$-3-\tfrac{20\pi}{3}\sin\left(\tfrac{\pi}{6}x\right)=0 \Leftrightarrow \sin\left(\tfrac{\pi}{6}x\right)=\tfrac{9}{20\pi} \Leftrightarrow \tfrac{\pi}{6}x\approx 0{,}14+n\cdot 2\pi \;\vee\; \tfrac{\pi}{6}x\approx\pi-0{,}14+n\cdot 2\pi \overset{:\frac{\pi}{6}}{\Leftrightarrow}$$

$$x\approx 0{,}27+12n \;\vee\; x\approx 5{,}73+12n \Leftrightarrow \underbrace{x\approx 0{,}27}_{\underbrace{0{,}27\cdot 30\approx 8}_{8/1-08}} \;\vee\; \underbrace{x\approx 5{,}73}_{\underbrace{0{,}73\cdot 30\approx 22}_{22/5-08}} \;\vee\; \underbrace{x\approx 12{,}27}_{8/1-09} \;\vee\; \underbrace{x\approx 17{,}73}_{22/5-09}$$



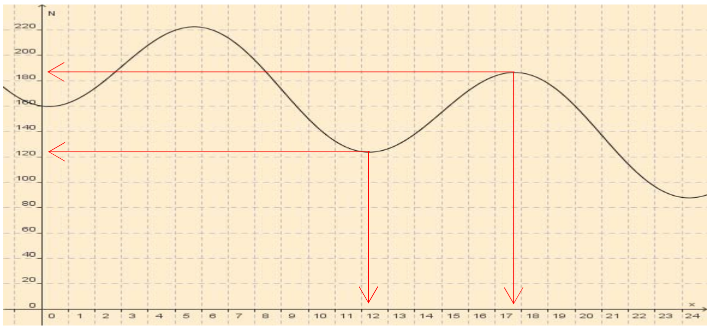
$$N(0{,}27)=200-3\cdot 0{,}27-40\cos\left(\tfrac{\pi}{6}\cdot 0{,}27\right)\approx 160$$

$$N(5{,}73)=200-3\cdot 5{,}73-40\cos\left(\tfrac{\pi}{6}\cdot 5{,}73\right)\approx 222$$

<u>Laveste medlemstall vinteren 2008 blir 160 den 8.januar.</u>

<u>Høyeste medlemstall i 2008 blir 222 den 22.mai.</u>

c)



d) Grafisk løst: $\underline{\underline{N(x)_{\text{min}} \approx N(12,3) \approx 124}}$ $\underline{\underline{N(x)_{\text{maks}} \approx N(17,7) \approx 186}}$

Fra fortegnsdiagrammet:

$N(x)_{\text{min}} \approx N(12,27) = 200 - 3 \cdot 12,27 - 40\cos\left(\frac{\pi}{6} \cdot 12,27\right) \approx 124$

$N(x)_{\text{maks}} \approx N(17,73) = 200 - 3 \cdot 17,73 - 40\cos\left(\frac{\pi}{6} \cdot 17,73\right) \approx 186$

<u>Laveste medlemstall vinteren 2009 blir 124 den 8.januar.</u>

<u>Høyeste medlemstall i 2009 blir 186 den 22.mai.</u>

e) $N'(x) = -3 + \frac{20\pi}{3}\sin\left(\frac{\pi}{6}x\right)$

Størst nedgang i medlemstallet gitt ved $N'(x)_{\text{min}} = -3 + \frac{20\pi}{3}(-1) \approx -24$

Dette skjer når $\sin\left(\frac{\pi}{6}x\right) = -1 \quad \Leftrightarrow \quad \frac{\pi}{6}x = \frac{3\pi}{2} + n \cdot 2\pi \quad \Leftrightarrow \quad x = 9 + 12n \quad \Leftrightarrow \quad x = 9 \quad \vee \quad x = 21$

<u>Medlemstallet minker maksimalt med 24 medlemmer i september 2008 og september 2009.</u>

Oppgave 3.65

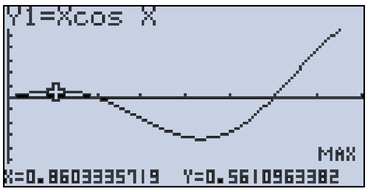
a) $f(x) = x\cos x \quad x \in [0, 2\pi\rangle$

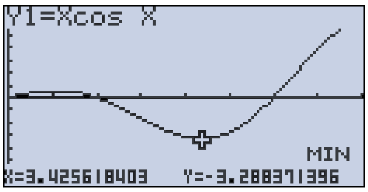
Nullpunkter : $f(x) = 0 \Rightarrow x\cos x = 0 \quad \Leftrightarrow \quad x = 0 \quad \vee \quad \cos x = 0 \quad \Leftrightarrow \quad x = 0 \quad \vee \quad x = \pm\frac{\pi}{2} + n \cdot 2\pi$

$\Leftrightarrow \quad \underline{\underline{x = 0}} \quad \vee \quad \underline{\underline{x = \frac{\pi}{2}}} \quad \vee \quad \underline{\underline{x = \frac{3\pi}{2}}}$

b) $f(x) = x\cos x \quad \Rightarrow \quad f'(x) = 1 \cdot \cos x + x \cdot (-\sin x) = \underline{\underline{\cos x - x\sin x}}$

c)

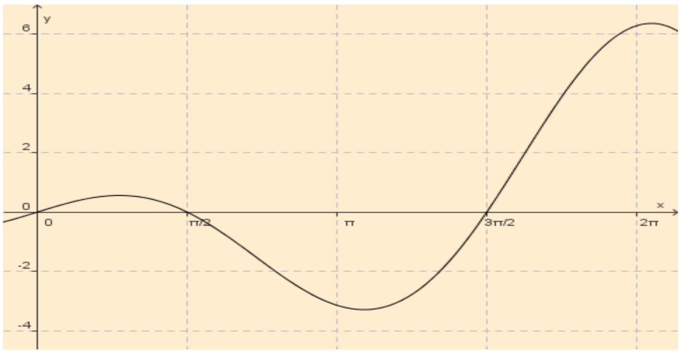




<u>Toppunkt $(0,86\ ,\ 0,56)$</u>

<u>Bunnpunkter $(3,43\ ,\ -3,29)$ og $(0,0)$</u>

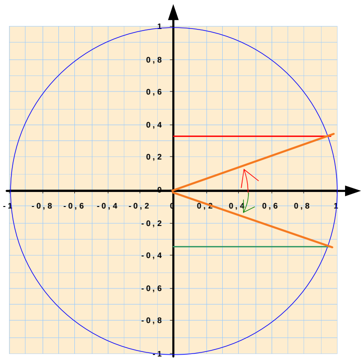
d)



3.7 Sum og differanse av vinkler

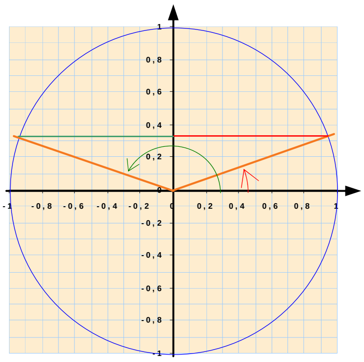
Oppgave 3.70

a)



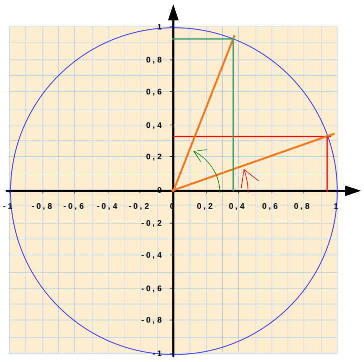
$$\sin(-v) = -\sin v = \underline{\underline{-\frac{1}{3}}}$$

b)



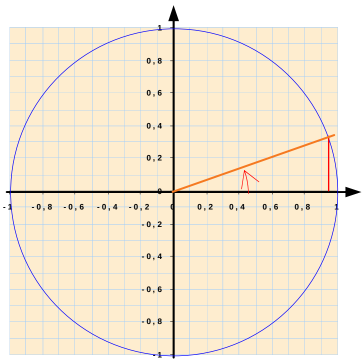
$$\sin(\pi - v) = \sin v = \underline{\underline{\frac{1}{3}}}$$

c)



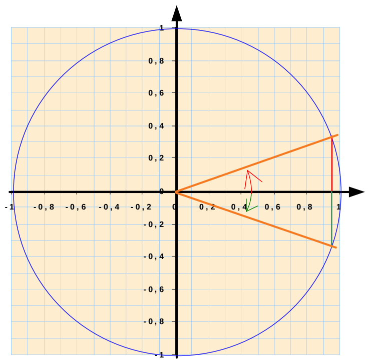
$$\cos\left(\frac{\pi}{2} - v\right) = \sin v = \underline{\underline{\frac{1}{3}}}$$

d)



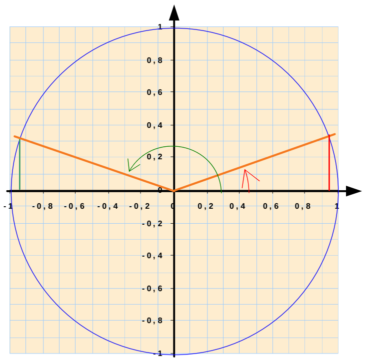
$$\cos v = \sqrt{1 - \left(\tfrac{1}{3}\right)^2} = \sqrt{1 - \tfrac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \underline{\underline{\frac{\sqrt{8}}{3}}}$$

e)



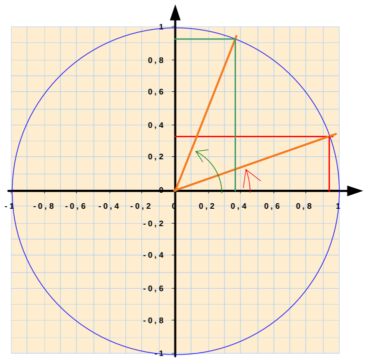
$$\cos(-v) = \cos v = \underline{\underline{\frac{\sqrt{8}}{3}}}$$

f)



$$\cos(\pi - v) = -\cos v = \underline{\underline{-\frac{\sqrt{8}}{3}}}$$

g)



$$\sin\left(\frac{\pi}{2} - v\right) = \cos v = \underline{\underline{\frac{\sqrt{8}}{3}}}$$

h)

$$\tan v = \frac{\sin v}{\cos v} = \frac{\frac{1}{3}}{\frac{\sqrt{8}}{3}} = \underline{\underline{\frac{1}{\sqrt{8}}}}$$

Oppgave 3.71

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cdot \cos 30° - \cos 45° \cdot \sin 30° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{2} \cdot \sqrt{3} - \sqrt{2}}{4} = \underline{\underline{\frac{\sqrt{6} - \sqrt{2}}{4}}}$$

$$\cos 15° = \cos(45° - 30°) = \cos 45° \cdot \cos 30° + \sin 45° \cdot \sin 30° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{2} \cdot \sqrt{3} + \sqrt{2}}{4} = \underline{\underline{\frac{\sqrt{6} + \sqrt{2}}{4}}}$$

$$\tan 15° = \frac{\sin 15°}{\cos 15°} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \underline{\underline{\frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}}}} = \frac{\cancel{\sqrt{2}} \cdot (\sqrt{3} - 1)}{\cancel{\sqrt{2}} \cdot (\sqrt{3} + 1)} = \underline{\underline{\frac{\sqrt{3} - 1}{\sqrt{3} + 1}}}$$

Oppgave 3.72

$$\sin 105° = \sin(45° + 60°) = \sin 45° \cdot \cos 60° - \cos 45° \cdot \sin 60° = \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$
$$= \underline{\underline{\frac{\sqrt{2} + \sqrt{6}}{4}}} = \underline{\underline{\frac{\sqrt{6} + \sqrt{2}}{4}}}$$

$$\cos 105° = \cos(45° + 60°) = \cos 45° \cdot \cos 60° - \sin 45° \cdot \sin 60° = \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$
$$= \underline{\underline{\frac{\sqrt{2} - \sqrt{6}}{4}}} = -\underline{\underline{\frac{\sqrt{6} - \sqrt{2}}{4}}}$$

$$\tan 105° = \frac{\sin 105°}{\cos 105°} = \frac{\frac{\sqrt{6}+\sqrt{2}}{4}}{-\frac{\sqrt{6}-\sqrt{2}}{4}} = -\underline{\underline{\frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}}}} = -\frac{\cancel{\sqrt{2}} \cdot (\sqrt{3} + 1)}{\cancel{\sqrt{2}} \cdot (\sqrt{3} - 1)} = -\underline{\underline{\frac{\sqrt{3} + 1}{\sqrt{3} - 1}}}$$

Oppgave 3.73

$$\sin 2v = \sin(v + v) = \sin v \cdot \cos v + \cos v \cdot \sin v = \underline{\underline{2 \sin v \cos v}}$$

$$\cos 2v = \cos(v + v) = \cos v \cdot \cos v - \sin v \cdot \sin v = \underline{\underline{\cos^2 v - \sin^2 v}}$$

Oppgave 3.74

$$\sqrt{2}\cos\left(3x+\tfrac{\pi}{4}\right)=\sqrt{2}\left[\cos 3x\cdot\cos\tfrac{\pi}{4}-\sin 3x\cdot\sin\tfrac{\pi}{4}\right]=\sqrt{2}\left[\cos 3x\cdot\tfrac{\sqrt{2}}{2}-\sin 3x\cdot\tfrac{\sqrt{2}}{2}\right]$$

$$=\sqrt{2}\cdot\tfrac{\sqrt{2}}{2}\left[\cos 3x-\sin 3x\right]=\underline{\underline{\cos 3x-\sin 3x}}$$

Oppgave 3.75

a)

$$2\sin\left(x-\tfrac{\pi}{4}\right)+2\sin\left(x+\tfrac{\pi}{4}\right)=2\cdot\left[\sin x\cdot\cos\tfrac{\pi}{4}-\cos x\cdot\sin\tfrac{\pi}{4}\right]+2\cdot\left[\sin x\cdot\cos\tfrac{\pi}{4}+\cos x\cdot\sin\tfrac{\pi}{4}\right]$$

$$=2\cdot\tfrac{\sqrt{2}}{2}\cdot\sin x-2\cdot\tfrac{\sqrt{2}}{2}\cdot\cos x+2\cdot\tfrac{\sqrt{2}}{2}\cdot\sin x+2\cdot\tfrac{\sqrt{2}}{2}\cdot\cos x$$

$$=\sqrt{2}\sin x-\sqrt{2}\cos x+\sqrt{2}\sin x+\sqrt{2}\cos x=\underline{\underline{2\sqrt{2}\sin x}}$$

b)

$$2\cos\left(x-\tfrac{\pi}{4}\right)-2\cos\left(x+\tfrac{\pi}{4}\right)=2\cdot\left[\cos x\cdot\cos\tfrac{\pi}{4}+\sin x\cdot\sin\tfrac{\pi}{4}\right]-2\cdot\left[\cos\cdot\cos\tfrac{\pi}{4}-\sin x\cdot\sin\tfrac{\pi}{4}\right]$$

$$=2\cdot\tfrac{\sqrt{2}}{2}\cdot\cos x+2\cdot\tfrac{\sqrt{2}}{2}\cdot\sin x-2\cdot\tfrac{\sqrt{2}}{2}\cdot\cos x+2\cdot\tfrac{\sqrt{2}}{2}\cdot\sin x$$

$$=\sqrt{2}\cos x+\sqrt{2}\sin x-\sqrt{2}\cos x+\sqrt{2}\sin x=\underline{\underline{2\sqrt{2}\sin x}}$$

Oppgave 3.76

$$\tan(u+v)=\frac{\sin(u+v)}{\cos(u+v)}=\frac{\sin u\cdot\cos v+\cos u\cdot\sin v}{\cos u\cdot\cos v-\sin u\cdot\sin v}=\frac{\dfrac{\sin u\cdot\cancel{\cos v}}{\cos u\cdot\cancel{\cos v}}+\dfrac{\cancel{\cos u}\cdot\sin v}{\cancel{\cos u}\cdot\cos v}}{\dfrac{\cancel{\cos u}\cdot\cancel{\cos v}}{\cancel{\cos u}\cdot\cancel{\cos v}}-\dfrac{\sin u\cdot\sin v}{\cos u\cdot\cos v}}=\underline{\underline{\frac{\tan u+\tan v}{1-\tan u\cdot\tan v}}}$$

3.8 Funksjonen $f(x) = a \cdot \sin kx + b \cdot \cos kx$

Oppgave 3.80

a) $2\sin\left(x+\frac{\pi}{3}\right) = 2\cdot\left(\sin x\cdot\cos\frac{\pi}{3} + \cos x\cdot\sin\frac{\pi}{3}\right) = 2\sin x\cdot\frac{1}{2} + 2\cos x\cdot\frac{\sqrt{3}}{2} = \underline{\underline{\sin x + \sqrt{3}\cos x}}$

b) $4\sin\left(2x-\frac{\pi}{4}\right) = 4\cdot\left(\sin 2x\cdot\cos\frac{\pi}{4} - \cos 2x\cdot\sin\frac{\pi}{4}\right) = 4\sin 2x\cdot\frac{\sqrt{2}}{2} - 4\cos 2x\cdot\frac{\sqrt{2}}{2}$

$= \underline{\underline{2\sqrt{2}\sin 2x - 2\sqrt{2}\cos 2x}}$

Oppgave 3.81

a) $\sin x + \cos x \quad \Rightarrow \quad A = \sqrt{1^2+1^2} = \sqrt{2} \quad \Rightarrow$

$\sqrt{2}\left(\sin x\cdot\frac{1}{\sqrt{2}} + \cos x\cdot\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\sin x\cdot\cos\varphi + \cos x\cdot\sin\varphi\right) = \sqrt{2}\sin\left(x+\varphi\right)$

$\cos\varphi = \frac{1}{\sqrt{2}} \quad \wedge \quad \sin\varphi = \frac{1}{\sqrt{2}} \quad \overset{\varphi\in\left[0,\frac{\pi}{2}\right]}{\Rightarrow} \quad \tan\varphi = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1 \quad \Leftrightarrow \quad \varphi = \frac{\pi}{4}$

$\sin x + \cos x = \underline{\underline{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}}$

b) $\sin x - \sqrt{3}\cos x \quad \Rightarrow \quad A = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2 \quad \Rightarrow$

$2\left(\sin x\cdot\frac{1}{2} - \cos x\cdot\frac{\sqrt{3}}{2}\right) = 2\left(\sin x\cdot\cos\varphi - \cos x\cdot\sin\varphi\right) = 2\sin\left(x-\varphi\right)$

$\cos\varphi = \frac{1}{2} \quad \wedge \quad \sin\varphi = \frac{\sqrt{3}}{2} \quad \overset{\varphi\in\left[0,\frac{\pi}{2}\right]}{\Rightarrow} \quad \tan\varphi = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad \Leftrightarrow \quad \varphi = \frac{\pi}{3}$

$\sin x + \cos x = \underline{\underline{2\sin\left(x-\frac{\pi}{3}\right)}}$

c) I uttrykket $-\sqrt{3}\sin 2x + \cos 2x$ er

$$A = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

Vi setter $-A$ utenfor en parentes. Det gir

$$-\sqrt{3}\sin 2x + \cos 2x = -2\left(\frac{\sqrt{3}}{2}\sin 2x - \frac{1}{2}\cos 2x\right) = -2\left(\sin 2x\cdot\frac{\sqrt{3}}{2} - \cos 2x\cdot\frac{1}{2}\right)$$

Nå velger vi $\varphi \in \left[0, \frac{\pi}{2}\right]$ slik at $\cos\varphi = \frac{\sqrt{3}}{2}$. Da blir automatisk $\sin\varphi = \frac{1}{2}$. Det gir $\varphi = \frac{\pi}{6}$.

$$-\sqrt{3}\sin 2x+\cos 2x=-2\cdot(\sin 2x\cdot\frac{\sqrt{3}}{2}-\cos 2x\cdot\frac{1}{2})$$
$$=-2\cdot(\sin 2x\cdot\cos\varphi-\cos 2x\cdot\sin\varphi)=-2\sin(2x-\varphi)$$
$$=\underline{\underline{-2\sin\left(2x-\frac{\pi}{6}\right)}}$$

Ovenfor brukte vi formelen for $\sin(u-v)$.

d)

$$3\sin\pi x-4\cos\pi x \;\Rightarrow\; A=\sqrt{3^2+(-4)^2}=\sqrt{25}=5 \;\Rightarrow$$
$$5\left(\sin\pi x\cdot\tfrac{3}{5}-\cos\pi x\cdot\tfrac{4}{5}\right)=5\left(\sin\pi x\cdot\cos\varphi-\cos\pi x\cdot\sin\varphi\right)=5\sin\left(\pi x-\varphi\right)$$
$$\cos\varphi=\tfrac{3}{5} \;\wedge\; \sin\varphi=\tfrac{4}{5} \;\Leftrightarrow\; \tan\varphi=\frac{\frac{4}{5}}{\frac{3}{5}}=\tfrac{4}{3} \;\overset{\varphi\in\left[0,\frac{\pi}{2}\right]}{\Leftrightarrow}\; \varphi=0,93$$

$$3\sin\pi x-4\cos\pi x=\underline{\underline{5\sin\left(\pi x-0,93\right)}}$$

Oppgave 3.82

a)

$$2\sqrt{3}\sin\pi x-2\cos\pi x \;\Rightarrow\; A=\sqrt{\left(2\sqrt{3}\right)^2+\left(-2\right)^2}=\sqrt{12+4}=\sqrt{16}=4 \;\Rightarrow$$
$$4\left(\sin\pi x\cdot\tfrac{2\sqrt{3}}{4}-\cos\pi x\cdot\tfrac{2}{4}\right)=4\left(\sin\pi x\cdot\cos\varphi-\cos\pi x\cdot\sin\varphi\right)=4\sin\left(\pi x-\varphi\right)$$
$$\cos\varphi=\tfrac{2\sqrt{3}}{4}=\tfrac{\sqrt{3}}{2} \;\wedge\; \sin\varphi=\tfrac{2}{4}=\tfrac{1}{2} \;\Leftrightarrow\; \tan\varphi=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}} \;\overset{\varphi\in\left[0,\frac{\pi}{2}\right]}{\Leftrightarrow}\; \varphi=\tfrac{\pi}{6}$$

$$f(x)=2\sqrt{3}\sin\pi x-2\cos\pi x=\underline{\underline{4\sin\left(\pi x-\tfrac{\pi}{6}\right)}}$$

b)

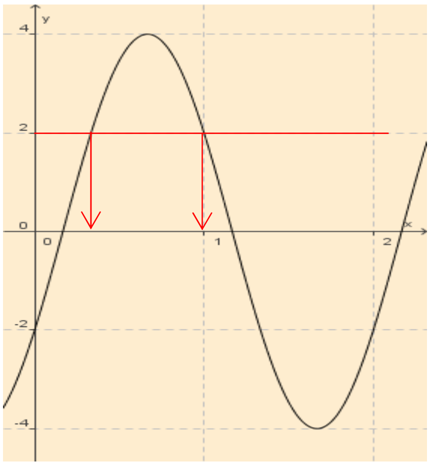
Amplitude: $\underline{\underline{A=4}}$ Periode: $\frac{2\pi}{\pi}=\underline{\underline{2}}$

c)

$$f(x)_{maks}=4\cdot 1=4 \text{ når } \sin\left(\pi x-\tfrac{\pi}{6}\right)=1 \;\Leftrightarrow\; \pi x-\tfrac{\pi}{6}=\tfrac{\pi}{2}+n\cdot 2\pi \;\overset{:\pi}{\Leftrightarrow}\; x-\tfrac{1}{6}=\tfrac{1}{2}+2n \;\Leftrightarrow$$
$$x=\tfrac{1}{2}+\tfrac{1}{6}+2n=\tfrac{2}{3}+2n \;\Leftrightarrow\; x=\tfrac{2}{3} \qquad \underline{\underline{\text{Toppunkt }\left(\tfrac{2}{3},4\right)}}$$

$$f(x)_{\min}=4\cdot(-1)=-4 \text{ når } \sin\left(\pi x-\tfrac{\pi}{6}\right)=-1 \;\Leftrightarrow\; \pi x-\tfrac{\pi}{6}=\tfrac{3\pi}{2}+n\cdot 2\pi \;\overset{:\pi}{\Leftrightarrow}\; x-\tfrac{1}{6}=\tfrac{3}{2}+2n \;\Leftrightarrow$$
$$x=\tfrac{3}{2}+\tfrac{1}{6}+2n=\tfrac{5}{3}+2n \;\Leftrightarrow\; x=\tfrac{5}{3} \qquad \underline{\underline{\text{Bunnpunkt }\left(\tfrac{5}{3},-4\right)}}$$

d)



e)

$f(x) = 2$ grafisk løst: $x = \frac{1}{3} \vee x = 1$

Løst ved regning:

$4\sin\left(\pi x - \frac{\pi}{6}\right) = 2 \Leftrightarrow \sin\left(\pi x - \frac{\pi}{6}\right) = \frac{1}{2} \Leftrightarrow$

$\pi x - \frac{\pi}{6} = \frac{\pi}{6} + n \cdot 2\pi \vee \pi x - \frac{\pi}{6} = \left(\pi - \frac{\pi}{6}\right) + n \cdot 2\pi \Leftrightarrow$

$\pi x - \frac{\pi}{6} = \frac{\pi}{6} + n \cdot 2\pi \vee \pi x - \frac{\pi}{6} = \frac{5\pi}{6} + n \cdot 2\pi \overset{:\pi}{\Leftrightarrow}$

$x - \frac{1}{6} = \frac{1}{6} + 2n \vee x - \frac{1}{6} = \frac{5}{6} + 2n \Leftrightarrow$

$x = \frac{1}{3} + 2n \vee x = 1 + 2n \Leftrightarrow \underline{\underline{x = \frac{1}{3}}} \vee \underline{\underline{x = 1}}$

3.9 Likningen $a \cdot \sin kx + b \cdot \cos kx = c$

Oppgave 3.90

a) $4\sin x + 4\cos x = 0 \quad x \in [-\pi, \pi]$

$$\overset{\cos x \neq 0}{\Leftrightarrow} \frac{4\sin x}{\cos x} + \frac{4\cos x}{\cos x} = \frac{0}{\cos x} \Leftrightarrow 4\tan x + 4 = 0 \Leftrightarrow \tan x = -1 \Leftrightarrow x = -\tfrac{\pi}{4} + n\cdot\pi$$

$$\Leftrightarrow \underline{\underline{x = -\tfrac{\pi}{4}}} \quad \vee \quad \underline{\underline{x = \tfrac{3\pi}{4}}}$$

b) $2\sin(2\pi x) - 2\sqrt{3}\cos(2\pi x) = 0 \quad x \in [0,1]$

$$\overset{\cos(2\pi x) \neq 0}{\Leftrightarrow} \frac{2\sin(2\pi x)}{\cos(2\pi x)} - \frac{2\sqrt{3}\cos(2\pi x)}{\cos(2\pi x)} = \frac{0}{\cos(2\pi x)} \Leftrightarrow 2\tan(2\pi x) - 2\sqrt{3} = 0 \Leftrightarrow$$

$$\tan(2\pi x) = \sqrt{3} \Leftrightarrow 2\pi x = \tfrac{\pi}{3} + n\cdot\pi \overset{:2\pi}{\Leftrightarrow} x = \tfrac{1}{6} + \tfrac{1}{2}n \Leftrightarrow \underline{\underline{x = \tfrac{1}{6}}} \quad \vee \quad \underline{\underline{x = \tfrac{2}{3}}}$$

c) $2\sin\left(\tfrac{\pi}{2}x\right) + 3\cos\left(\tfrac{\pi}{2}x\right) = 0 \quad x \in [-2, 2]$

$$\overset{\cos\left(\frac{\pi}{2}x\right) \neq 0}{\Leftrightarrow} \frac{2\sin\left(\frac{\pi}{2}x\right)}{\cos\left(\frac{\pi}{2}x\right)} + \frac{3\cos\left(\frac{\pi}{2}x\right)}{\cos\left(\frac{\pi}{2}x\right)} = \frac{0}{\cos\left(\frac{\pi}{2}x\right)} \Leftrightarrow 2\tan\left(\tfrac{\pi}{2}x\right) + 3 = 0 \Leftrightarrow$$

$$\tan\left(\tfrac{\pi}{2}x\right) = -\tfrac{3}{2} \Leftrightarrow \tfrac{\pi}{2}x \approx -0{,}98 + n\cdot\pi \overset{:\frac{\pi}{2}}{\Leftrightarrow} x \approx -0{,}63 + 2n \Leftrightarrow \underline{\underline{x \approx -0{,}63}} \quad \vee \quad \underline{\underline{x \approx 1{,}37}}$$

Oppgave 3.91

a) $\sin x + \cos x \Rightarrow A = \sqrt{1^2 + 1^2} = \sqrt{2} \Rightarrow$

$$\sqrt{2}\left(\sin x \cdot \tfrac{1}{\sqrt{2}} + \cos x \cdot \tfrac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\sin x \cdot \cos\varphi + \cos x \cdot \sin\varphi\right) = \sqrt{2}\sin(x + \varphi)$$

$$\cos\varphi = \tfrac{1}{\sqrt{2}} \quad \wedge \quad \sin\varphi = \tfrac{1}{\sqrt{2}} \overset{\varphi \in \left[0, \frac{\pi}{2}\right]}{\Rightarrow} \tan\varphi = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1 \Leftrightarrow \varphi = \tfrac{\pi}{4}$$

$$\sin x + \cos x = \sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right)$$

$\sin x + \cos x = 1 \quad x \in [0, 2\pi]$

$$\Leftrightarrow \sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right) = 1 \Leftrightarrow \sin\left(x + \tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}} \Leftrightarrow x + \tfrac{\pi}{4} = \tfrac{\pi}{4} + n\cdot 2\pi \quad \vee \quad x + \tfrac{\pi}{4} = \left(\pi - \tfrac{\pi}{4}\right) + n\cdot 2\pi$$

$$\Leftrightarrow x = n\cdot 2\pi \quad \vee \quad x = \tfrac{\pi}{2} + n\cdot 2\pi \Leftrightarrow \underline{\underline{x = 0}} \quad \vee \quad \underline{\underline{x = \tfrac{\pi}{2}}} \quad \vee \quad \underline{\underline{x = 2\pi}}$$

b) I likningen

$$\sqrt{3}\sin 2x - \cos 2x = \sqrt{2}\,, \qquad x \in [0, \pi]$$

er

$$A = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$

Vi dividerer med $A = 2$ på begge sidene av likhetstegnet.

$$\sqrt{3}\sin 2x - \cos 2x = \sqrt{2}$$
$$\frac{\sqrt{3}}{2}\sin 2x - \frac{1}{2}\cos 2x = \frac{\sqrt{2}}{2}$$
$$\sin 2x \cdot \frac{\sqrt{3}}{2} - \cos 2x \cdot \frac{1}{2} = \frac{\sqrt{2}}{2}$$

Nå velger vi $\varphi \in \left[0, \frac{\pi}{2}\right]$ slik at $\cos\varphi = \frac{\sqrt{3}}{2}$. Da blir automatisk $\sin\varphi = \frac{1}{2}$. Det gir $\varphi = \frac{\pi}{6}$.

Likningen blir

$$\sin 2x \cdot \cos\varphi - \cos 2x \cdot \sin\varphi = \frac{\sqrt{2}}{2}$$
$$\sin(2x - \varphi) = \frac{\sqrt{2}}{2}$$
$$2x - \varphi = \frac{\pi}{4} \quad \text{eller} \quad 2x - \varphi = \frac{3\pi}{4}$$
$$2x = \frac{\pi}{4} + \varphi = \frac{\pi}{4} + \frac{\pi}{6} \quad \text{eller} \quad 2x = \frac{3\pi}{4} + \varphi = \frac{3\pi}{4} + \frac{\pi}{6}$$
$$2x = \frac{5\pi}{12} \quad \text{eller} \quad 2x = \frac{11\pi}{12}$$
$$\underline{\underline{x = \frac{5\pi}{24} \quad \text{eller} \quad x = \frac{11\pi}{24}}}$$

c) I likningen

$$-3\sin\frac{\pi}{2}x+4\cos\frac{\pi}{2}x=1, \qquad x\in[-2,\,2]$$

er

$$A=\sqrt{(-3)^2+4^2}=\sqrt{25}=5$$

Vi dividerer med $A=-5$ på begge sidene av likhetstegnet.

$$\frac{3}{5}\sin\frac{\pi}{2}x-\frac{4}{5}\cos\frac{\pi}{2}x=-\frac{1}{5}$$
$$\sin\frac{\pi}{2}x\cdot\frac{3}{5}-\cos\frac{\pi}{2}x\cdot\frac{4}{5}=-\frac{1}{5}$$

Nå velger vi $\varphi\in\left[0,\frac{\pi}{2}\right]$ slik at $\cos\varphi=\frac{3}{5}$. Da blir automatisk $\sin\varphi=\frac{4}{5}$. Det gir $\varphi=0{,}927$.
Likningen blir

$$\sin\frac{\pi}{2}x\cdot\cos\varphi-\cos\frac{\pi}{2}x\cdot\sin\varphi=-\frac{1}{5}$$
$$\sin(\frac{\pi}{2}x-\varphi)=-\frac{1}{5}$$

Ovenfor brukte vi formelen for sin(*u* – *v*). Ved hjelp av lommeregneren og enhetssirkelen finner vi at

$$\frac{\pi}{2}x-\varphi=-0{,}201 \text{ eller } \frac{\pi}{2}x-\varphi=-2{,}940$$
$$\frac{\pi}{2}x=\varphi-0{,}201 \text{ eller } \frac{\pi}{2}x=\varphi-2{,}940$$
$$\frac{\pi}{2}x=0{,}927-0{,}201 \text{ eller } \frac{\pi}{2}x=0{,}927-2{,}940$$
$$\frac{\pi}{2}x=0{,}726 \text{ eller } \frac{\pi}{2}x=-2{,}013$$
$$x=0{,}726\cdot\frac{2}{\pi} \text{ eller } x=-2{,}013\cdot\frac{2}{\pi}$$
$$\underline{\underline{x=0{,}46 \quad \text{eller} \quad x=-1{,}28}}$$

d) $2\sin\frac{\pi}{6}x - 3\cos\frac{\pi}{6}x \;\Rightarrow A = \sqrt{(2)^2 + (-3)^2} = \sqrt{13} \;\Rightarrow$

$\sqrt{13}\left(\sin\frac{\pi}{6}x\cdot\frac{2}{\sqrt{13}} - \cos\frac{\pi}{6}x\cdot\frac{3}{\sqrt{13}}\right) = \sqrt{13}\left(\sin\frac{\pi}{6}x\cdot\cos\varphi - \cos\frac{\pi}{6}x\cdot\sin\varphi\right) = \sqrt{13}\sin(x-\varphi)$

$\cos\varphi = \frac{2}{\sqrt{13}} \quad\wedge\quad \sin\varphi = \frac{3}{\sqrt{13}} \quad\Leftrightarrow\quad \tan\varphi = \frac{\frac{3}{\sqrt{13}}}{\frac{2}{\sqrt{13}}} = \frac{3}{2} \quad\overset{\varphi\in\left[0,\frac{\pi}{2}\right]}{\Leftrightarrow}\quad \varphi \approx 0{,}98$

$2\sin\frac{\pi}{6}x - 3\cos\frac{\pi}{6}x \approx \sqrt{13}\sin\left(\frac{\pi}{6}x - 0{,}98\right)$

$2\sin\frac{\pi}{6}x - 3\cos\frac{\pi}{6}x = 4, \quad x\in[0,12]$

$\Leftrightarrow \quad \sqrt{13}\sin\left(\frac{\pi}{6}x - 0{,}98\right) = 4 \quad\Leftrightarrow\quad \sin\left(\frac{\pi}{6}x - 0{,}98\right) = \frac{4}{\sqrt{13}} \approx 1{,}1094$

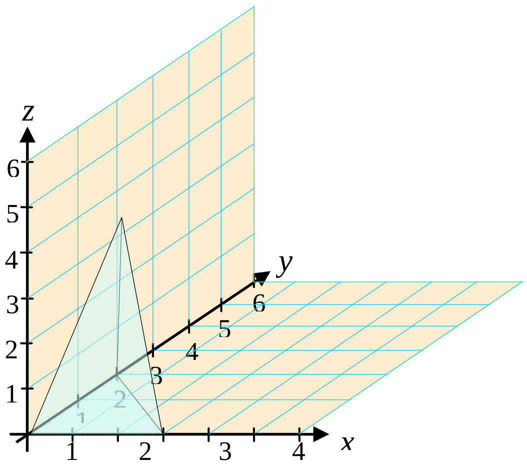
$\sin v \leq 1$ for alle verdier av v.

<u>Likningen har ingen løsning.</u>

4.1 Romkoordinater

Oppgave 4.10

a)



b) Grunnflata ABC ligger i xy-planet, derfor kan man se bort fra z-koordinatene.

$$\overrightarrow{AB} = [3-0, 0-0] = [3, 0] \quad \Rightarrow \quad |\overrightarrow{AB}| = \sqrt{3^2 + 0^2} = 3$$

$$\overrightarrow{AC} = [0-0, 2-0] = [0, 2] \quad \Rightarrow \quad |\overrightarrow{AC}| = \sqrt{0^2 + 2^2} = 2$$

$$\overrightarrow{AB} \perp \overrightarrow{AC} \quad \text{fordi } [3, 0] \cdot [0, 2] = 0 + 0 = 0$$

$$A_{\text{grunnflate}} = \frac{|\overrightarrow{AB}| \cdot |\overrightarrow{AC}|}{2} = \frac{3 \cdot 2}{2} = \underline{\underline{3}}$$

c) Toppunktet T i pyramiden ligger 4 enheter over grunnflata $\Rightarrow$ $\underline{\underline{\text{Høyden i pyramiden er 4.}}}$

d) $$V_{\text{pyramide}} = \frac{G \cdot h}{3} = \frac{3 \cdot 4}{3} = \underline{\underline{4}}$$

Oppgave 4.11

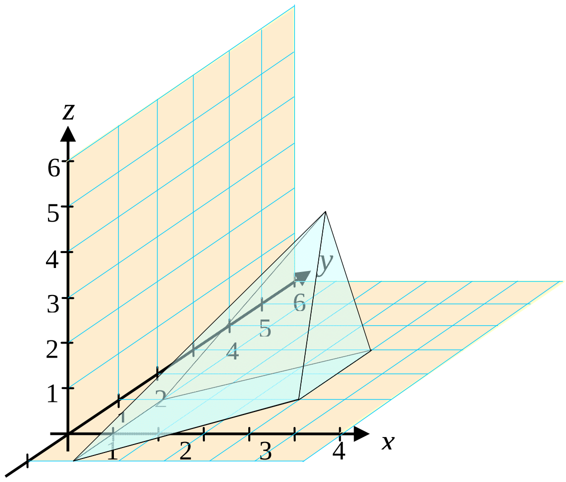
Avstanden til xy-planet er 2 $\Rightarrow$ $z = 2$

Avstanden til xz-planet er 3 $\Rightarrow$ $y = 3$ $\quad \Leftrightarrow \quad$ $\underline{\underline{A(4, 3, 2)}}$

Avstanden til yz-planet er 4 $\Rightarrow$ $x = 4$

Oppgave 4.12

a)



b) Grunnflata *ABCD* ligger i *xy*-planet, derfor kan man se bort fra *z*-koordinatene.

$\overrightarrow{AB} = [4-1, 1-(-1)] = [3, 2] \quad \Rightarrow \quad |\overrightarrow{AB}| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\overrightarrow{AD} = [1-1, 1-(-1)] = [0, 2] \quad \Rightarrow \quad |\overrightarrow{AD}| = \sqrt{0^2 + 2^2} = 2$

$$\cos\angle(\overrightarrow{AB}, \overrightarrow{AD}) = \frac{\overrightarrow{AB}\cdot\overrightarrow{AD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{AD}|} = \frac{[3,2]\cdot[0,2]}{\sqrt{13}\cdot 2} = \frac{0+4}{\sqrt{13}\cdot 2} = \frac{4}{\sqrt{13}\cdot 2} = \frac{2}{\sqrt{13}}$$

$\Rightarrow \quad \angle(\overrightarrow{AB}, \overrightarrow{AD}) \approx 56{,}31°$

$$A_{\text{grunnflate}} = \frac{|\overrightarrow{AB}|\cdot|\overrightarrow{AD}|\cdot\sin\angle(\overrightarrow{AB}, \overrightarrow{AD})}{2}\cdot 2 = \sqrt{13}\cdot 2\cdot\sin 56{,}31° = \underline{\underline{6}}$$

c) $V_{pyramide} = \dfrac{G\cdot h}{3} = \dfrac{6\cdot 3}{3} = \underline{\underline{6}}$

Oppgave 4.13

a) Punktene $A(1, 2, 1)$, $B(3, 2, 3)$ og $C(-1, 2, 3)$ har alle andrekoordinat lik 2.
Dette betyr at disse punktene ligger i et plan parallelt med *xz*-planet.
Da kan vi se bort fra andre.koordinatene og si at:

$\overrightarrow{AB} = [3-1, 3-1] = [2, 2] \quad \overrightarrow{BC} = [-1-3, 3-3] = [-4, 0] \quad \overrightarrow{AC} = [-1-1, 3-1] = [-2, 2]$

ΔABC rettvinklet $\Leftrightarrow \overrightarrow{AB}\cdot\overrightarrow{AC} = 0 \;\vee\; \overrightarrow{AB}\cdot\overrightarrow{BC} = 0 \;\vee\; \overrightarrow{AC}\cdot\overrightarrow{BC} = 0$

$\overrightarrow{AB}\cdot\overrightarrow{AC} = [2, 2]\cdot[-2, 2] = -4+4 = 0 \quad \Rightarrow \quad \angle A = 90° \qquad \underline{\underline{\Delta ABC \text{ er rettvinklet.}}}$

b) $\left|\overrightarrow{AB}\right| = \sqrt{2^2 + 2^2} = \sqrt{8}$ $\left|\overrightarrow{AC}\right| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$ $\Rightarrow$ $A_{\text{grunnflate}} = \dfrac{\sqrt{8} \cdot \sqrt{8}}{2} = 4$

Toppunktet D har 2.koordinat lik 5, og ligger dermed 3 enheter fra planet grunnflata ligger i.
$\Rightarrow h_{\text{pyramide}} = 3$

$$V_{\text{pyramide}} = \frac{4 \cdot 3}{3} = \underline{\underline{4}}$$

4.2 Vektorer i rommet

Oppgave 4.20

a) $2\left(\vec{u}+2\vec{v}\right)+4\left(\frac{1}{2}\vec{u}-\vec{v}\right)=2\vec{u}+4\vec{v}+2\vec{u}-4\vec{v}=\underline{\underline{4\vec{u}}}$

b) $\frac{1}{3}\left(2\vec{u}-4\vec{v}\right)-\frac{1}{2}\left(3\vec{u}-5\vec{v}\right)=\frac{2}{3}\vec{u}-\frac{4}{3}\vec{v}-\frac{3}{2}\vec{u}+\frac{5}{2}\vec{v}=\underline{\underline{-\frac{5}{6}\vec{u}+\frac{7}{6}\vec{v}}}$

c) $\frac{3}{2}\left(4\vec{u}-\vec{v}\right)-\frac{3}{4}\left(8\vec{u}-2\vec{v}\right)=6\vec{u}-\frac{3}{2}\vec{v}-6\vec{u}+\frac{3}{2}\vec{v}=\underline{\underline{\vec{0}}}$

Oppgave 4.21

a) $\vec{u}=6\vec{a}+8\vec{b} \qquad \vec{v}=9\vec{a}+12\vec{b} \qquad \vec{u}\parallel\vec{v} \iff \vec{u}=t\cdot\vec{v}$

$6\vec{a}+8\vec{b}=t\cdot\left(9\vec{a}+12\vec{b}\right) \iff 6=9t \wedge 8=12t \iff t=\frac{6}{9}=\frac{2}{3} \wedge t=\frac{8}{12}=\frac{2}{3}$

$\vec{u}=\frac{2}{3}\cdot\vec{v} \quad\Rightarrow\quad \underline{\underline{\vec{u}\parallel\vec{v}}}$

b) $\vec{u}=8\vec{a}-6\vec{b} \qquad \vec{v}=-12\vec{a}+9\vec{b}$

$8\vec{a}-6\vec{b}=t\cdot\left(-12\vec{a}+9\vec{b}\right) \iff 8=-12t \wedge -6=9t \iff t=\frac{8}{-12}=-\frac{2}{3} \wedge t=\frac{-6}{9}=-\frac{2}{3}$

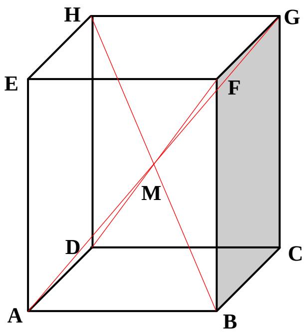
$\vec{u}=-\frac{2}{3}\cdot\vec{v} \quad\Rightarrow\quad \underline{\underline{\vec{u}\parallel\vec{v}}}$

c) $\vec{u}=-36\vec{a}+42\vec{b} \qquad \vec{v}=55\vec{a}-65\vec{b}$

$-36\vec{a}+42\vec{b}=t\cdot\left(55\vec{a}-65\vec{b}\right) \iff -36=55t \wedge 42=-65t \iff t=\frac{-36}{55}=-\frac{36}{55} \wedge t=\frac{-42}{65}=-\frac{42}{65}$

$\Rightarrow \quad \underline{\underline{\vec{u}\nparallel\vec{v}}}$

Oppgave 4.22



M er midtpunktet på CE $\Leftrightarrow$ $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CE}$

$$\begin{aligned}\overrightarrow{AM} &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CM} = \overrightarrow{AB} + \overrightarrow{BC} + \tfrac{1}{2}\overrightarrow{CE} \\ &= \overrightarrow{AB} + \overrightarrow{BC} + \tfrac{1}{2}\left(\overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AE}\right) \\ &= \overrightarrow{AB} + \overrightarrow{BC} + \tfrac{1}{2}\left(-\overrightarrow{BC} - \overrightarrow{AB} + \left(\overrightarrow{AG} + \overrightarrow{GH} + \overrightarrow{HE}\right)\right) \\ &= \overrightarrow{AB} + \overrightarrow{BC} + \tfrac{1}{2}\left(-\overrightarrow{BC} - \overrightarrow{AB} + \left(\overrightarrow{AG} - \overrightarrow{AB} - \overrightarrow{BC}\right)\right) \\ &= \overrightarrow{AB} + \overrightarrow{BC} + \tfrac{1}{2}\left(-2\overrightarrow{BC} - 2\overrightarrow{AB} + \overrightarrow{AG}\right) \\ &= \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{BC} - \overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{AG}\end{aligned}$$

$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AG}$ $\Leftrightarrow$ M er midtpunktet på AG.

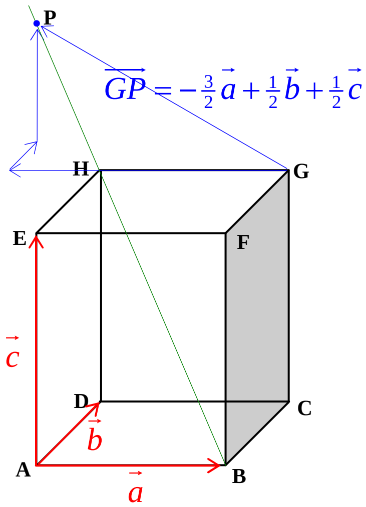
$$\begin{aligned}\overrightarrow{BM} &= \overrightarrow{BC} + \overrightarrow{CM} = \overrightarrow{BC} + \tfrac{1}{2}\overrightarrow{CE} \\ &= \overrightarrow{BC} + \tfrac{1}{2}\left(\overrightarrow{CB} + \overrightarrow{BH} + \overrightarrow{HE}\right) \\ &= \overrightarrow{BC} + \tfrac{1}{2}\left(-\overrightarrow{BC} + \overrightarrow{BH} - \overrightarrow{BC}\right) \\ &= \overrightarrow{BC} + \tfrac{1}{2}\left(-2\overrightarrow{BC} + \overrightarrow{BH}\right) \\ &= \overrightarrow{BC} - \overrightarrow{BC} + \tfrac{1}{2}\overrightarrow{BH} = \tfrac{1}{2}\overrightarrow{BH}\end{aligned}$$

$$\begin{aligned}\overrightarrow{DM} &= \overrightarrow{DC} + \overrightarrow{CM} = \overrightarrow{DC} + \tfrac{1}{2}\overrightarrow{CE} \\ &= \overrightarrow{DC} + \tfrac{1}{2}\left(\overrightarrow{CD} + \overrightarrow{DF} + \overrightarrow{FE}\right) \\ &= \overrightarrow{DC} + \tfrac{1}{2}\left(-\overrightarrow{DC} + \overrightarrow{DF} - \overrightarrow{DC}\right) \\ &= \overrightarrow{DC} + \tfrac{1}{2}\left(-2\overrightarrow{DC} + \overrightarrow{DF}\right) \\ &= \overrightarrow{DC} - \overrightarrow{DC} + \tfrac{1}{2}\overrightarrow{DF} = \tfrac{1}{2}\overrightarrow{DF}\end{aligned}$$

$\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BH}$ $\Leftrightarrow$ M er midtpunktet på BH.

$\overrightarrow{DM} = \frac{1}{2}\overrightarrow{DF}$ $\Leftrightarrow$ M er midtpunktet på DF.

Oppgave 4.23



$$\overrightarrow{BH} = -\vec{a} + \vec{b} + \vec{c}$$

$$\overrightarrow{BP} = \overrightarrow{BC} + \overrightarrow{CG} + \overrightarrow{GP} = \vec{b} + \vec{c} + \left(-\tfrac{3}{2}\vec{a} + \tfrac{1}{2}\vec{b} + \tfrac{1}{2}\vec{c}\right)$$

$$= \vec{b} + \vec{c} - \tfrac{3}{2}\vec{a} + \tfrac{1}{2}\vec{b} + \tfrac{1}{2}\vec{c} = -\tfrac{3}{2}\vec{a} + \tfrac{3}{2}\vec{b} + \tfrac{3}{2}\vec{c}$$

$$B, H \text{ og } P \text{ på linje} \quad \Leftrightarrow \quad \overrightarrow{BH} \parallel \overrightarrow{BP} \quad \Leftrightarrow \quad \overrightarrow{BH} = t \cdot \overrightarrow{BP}$$

$$-\vec{a} + \vec{b} + \vec{c} = t \cdot \left(-\tfrac{3}{2}\vec{a} + \tfrac{3}{2}\vec{b} + \tfrac{3}{2}\vec{c}\right) \quad \Leftrightarrow$$

$$-\tfrac{3}{2}t = -1 \ \wedge \ \tfrac{3}{2}t = 1 \ \wedge \ \tfrac{3}{2}t = 1 \quad \Leftrightarrow \quad t = \tfrac{2}{3} \ \wedge \ t = \tfrac{2}{3} \ \wedge \ t = \tfrac{2}{3}$$

$$\overrightarrow{BH} = \tfrac{2}{3} \cdot \overrightarrow{BP} \quad \Rightarrow \quad \underline{\underline{B, H \text{ og } P \text{ ligger på linje.}}}$$

4.3 Vektorkoordinater

Oppgave 4.30

a) $\vec{u} = [1, -2, 3] \qquad \vec{v} = [3, 1, 1]$

$$\vec{u} + \vec{v} = [1, -2, 3] + [3, 1, 1] = [1+3, -2+1, 3+1] = \underline{\underline{[4, -1, 4]}}$$

b) $\vec{u} - \vec{v} = [1, -2, 3] - [3, 1, 1] = [1-3, -2-1, 3-1] = \underline{\underline{[-2, -3, 2]}}$

c) $2\vec{u} + 3\vec{v} = 2 \cdot [1, -2, 3] + 3 \cdot [3, 1, 1] = [2, -4, 6] + [9, 3, 3] = [2+9, -4+3, 6+3] = \underline{\underline{[11, -1, 9]}}$

d) $3\vec{u} - 4\vec{v} = 3 \cdot [1, -2, 3] - 4 \cdot [3, 1, 1] = [3, -6, 9] - [12, 4, 4] = [3-12, -6-4, 9-4] = \underline{\underline{[-9, -10, 5]}}$

Oppgave 4.31

a) $[-12, 18, 6] = t \cdot [18, -27, -9] \quad \Leftrightarrow \quad 18t = -12 \quad \wedge \quad -27t = 18 \quad \wedge \quad -9t = 6 \quad \Leftrightarrow$

$t = \frac{-12}{18} = -\frac{2}{3} \quad \wedge \quad t = \frac{18}{-27} = -\frac{2}{3} \quad \wedge \quad t = \frac{6}{-9} = -\frac{2}{3}$

$$[-12, 18, 6] = -\tfrac{2}{3} \cdot [18, -27, -9] \quad \Rightarrow \quad \underline{\underline{[-12, 18, 6] \parallel [18, -27, -9]}}$$

b) $[6, 9, -15] = t \cdot [8, 12, -21] \quad \Leftrightarrow \quad 8t = 6 \quad \wedge \quad 12t = 9 \quad \wedge \quad -21t = -15 \quad \Leftrightarrow$

$t = \frac{6}{8} = \frac{3}{4} \quad \wedge \quad t = \frac{9}{12} = \frac{3}{4} \quad \wedge \quad t = \frac{-15}{-21} = \frac{5}{7}$

Ingen slik t-verdi finnes og derfor er ikke vektorene parallelle. $\underline{\underline{[6, 9, -15] \nparallel [8, 12, -21]}}$

Oppgave 4.32

a) $\vec{a}=[1,0,1] \qquad \vec{b}=[-2,1,3] \qquad \vec{c}=[-5,0,-5]$

$\vec{u}=[1,2,-1]+t\cdot[3,1,2]=[1+3t,2+t,-1+2t]$

$[1+3t,2+t,-1+2t]=[1,0,1] \Leftrightarrow 1+3t=1 \wedge 2+t=0 \wedge -1+2t=1 \Leftrightarrow$

$t=0 \wedge t=-2 \wedge t=1 \Rightarrow \vec{u}\neq\vec{a}$

$[1+3t,2+t,-1+2t]=[-2,1,3] \Leftrightarrow 1+3t=-2 \wedge 2+t=1 \wedge -1+2t=3 \Leftrightarrow$

$t=-1 \wedge t=-1 \wedge t=2 \Rightarrow \vec{u}\neq\vec{b}$

$[1+3t,2+t,-1+2t]=[-5,0,-5] \Leftrightarrow 1+3t=-5 \wedge 2+t=0 \wedge -1+2t=-5 \Leftrightarrow$

$t=-2 \wedge t=-2 \wedge t=-2 \Rightarrow \underline{\underline{\vec{u}=\vec{c} \text{ når } t=-2}}$

b) $[1+3t,2+t,-1+2t]=s\cdot[1,0,1] \Leftrightarrow 1+3t=s \wedge 2+t=0 \wedge -1+2t=s \Leftrightarrow$

$t=-2 \wedge 1+3t=-1+2t \Leftrightarrow t=-2 \wedge t=-2 \Rightarrow \underline{\underline{\vec{u}\parallel\vec{a} \text{ når } t=-2}}$

Oppgave 4.33

a) Gitt punktene $O(0,0,0)$, $A(2,1,0)$, $B(3,3,1)$

$\underline{\underline{\overrightarrow{OA}=[2,1,0]}} \qquad \underline{\underline{\overrightarrow{OB}=[3,3,1]}} \qquad \overrightarrow{AB}=[3-2,3-1,1-0]=\underline{\underline{[1,2,1]}}$

b) Firkant $OABC$ et parallellogram $\Leftrightarrow \overrightarrow{OC}=\overrightarrow{AB} \Rightarrow \overrightarrow{OC}=[1,2,1] \Leftrightarrow \underline{\underline{C(1,2,1)}}$

Oppgave 4.34

Gitt punktene $A(-1,2,3)$ og $B(3,-4,1) \Rightarrow \overrightarrow{AB}=[3-(-1),-4-2,1-3]=[4,-6,-2]$

M er midtpunktet på $AB \Leftrightarrow \overrightarrow{AM}=\frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=[-1,2,3]+[2,-3,-1]=[-1+2,2+(-3),3+(-1)]=[1,-1,2] \Rightarrow \underline{\underline{M(1,-1,2)}}$

Oppgave 4.35

Gitt punktene $A(2,2,1)$, $B(3,4,2)$ og $C(1,-4,0)$ $\Rightarrow$ $\overrightarrow{AB}=[3-2,4-2,2-1]=[1,2,1]$

D ligger på i xz-planet $\Rightarrow$ $D(x,0,z)$ $\overrightarrow{CD}=\left[x-1,0-(-4),z-0\right]=[x-1,4,z]$

$CD \parallel AB \Leftrightarrow \overrightarrow{CD}=t\cdot\overrightarrow{AB} \Rightarrow [x-1,4,z]=t\cdot[1,2,1] \Leftrightarrow x-1=t \wedge 2t=4 \wedge z=t \Leftrightarrow$
$t=2 \wedge z=2 \wedge x=2+1=3 \Rightarrow \underline{\underline{D(3,0,2)}}$

Oppgave 4.36

a) Gitt punktene $A(1,2,1)$, $B(3,1,2)$, $C(4,2,0)$
$\Rightarrow \overrightarrow{AC}=[4-1,2-2,0-1]=[3,0,-1]$ $\overrightarrow{BC}=[4-3,2-1,0-2]=[1,1,-2]$

$\overrightarrow{BD}=2\overrightarrow{AC}+\overrightarrow{BC}=2\cdot[3,0,-1]+[1,1,-2]=[6,0,-2]+[1,1,-2]=\left[6+1,0+1,-2+(-2)\right]=[7,1,-4]$
$\overrightarrow{OD}=\overrightarrow{OB}+\overrightarrow{BD}=[3,1,2]+[7,1,-4]=\left[3+7,1+1,2+(-4)\right]=[10,2,-2]$
$\Rightarrow \underline{\underline{D(10,2,-2)}}$

b) $\overrightarrow{AD}=[10-1,2-2,-2-1]=[9,0,-3]$

A, C og D på linje $\Leftrightarrow \overrightarrow{AC}\parallel\overrightarrow{AD} \Leftrightarrow \overrightarrow{AC}=t\cdot\overrightarrow{AD}$
$[3,0,-1]=t\cdot[9,0,-3] \Leftrightarrow 9t=3 \wedge -3t=-1 \Leftrightarrow$
$t=\frac{3}{9}=\frac{1}{3} \wedge t=\frac{-1}{-3}=\frac{1}{3}$

$[3,0,-1]=\frac{1}{3}\cdot[9,0,-3] \Rightarrow$ $\underline{\underline{A, C \text{ og } D \text{ ligger på linje}}}$

Oppgave 4.37

Gitt punktene $A(2,2,-1)$, $B(1,0,2)$, $C(3,-1,0)$ og D plassert slik at $\overrightarrow{BD}=\overrightarrow{AB}+s\cdot\overrightarrow{BC}$

$\overrightarrow{AB}=\left[1-2,0-2,2-(-1)\right]=[-1,-2,3]$

$\overrightarrow{BC}=[3-1,-1-0,0-2]=[2,-1,-2]$

$\overrightarrow{AC}=\left[3-2,-1-2,0-(-1)\right]=[1,-3,1]$

$\overrightarrow{BD}=[-1,-2,3]+s\cdot[2,-1,-2]=[-1+2s,-2-s,3-2s]$

$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}=[-1,-2,3]+[-1+2s,-2-s,3-2s]=\left[-1+(-1+2s),-2+(-2-s),3+(3-2s)\right]=$

$A,\ C$ og D på linje $\Leftrightarrow$ $\overrightarrow{AC}=t\cdot\overrightarrow{AD}$

$[1,-3,1]=t\cdot[2s-2,-s-4,6-2s]$ $\Leftrightarrow$ $t\cdot(2s-2)=1$ $\wedge$ $t\cdot(-s-4)=-3$ $\wedge$ $t\cdot(6-2s)=1$ $\Leftrightarrow$

$t\cdot(2s-2)=t\cdot(6-2s)$ $\Leftrightarrow$ $2s-2=6-2s$ $\Leftrightarrow$ $4s=8$ $\Leftrightarrow$ $s=2$

Når $s=2$ ligger punktene A, C og D på linje.

4.4 Lengden av en vektor

Oppgave 4.40

a) $\vec{a}=[2,3,1] \quad \Rightarrow |\vec{a}|=\sqrt{2^2+3^2+1^2}=\underline{\underline{\sqrt{14}}}$

b) $\vec{a}=[3,-1,2] \quad \Rightarrow |\vec{a}|=\sqrt{3^2+(-1)^2+2^2}=\underline{\underline{\sqrt{14}}}$

Oppgave 4.41

a) $A(1,2,-2)$ og $B(0,1,3) \quad \Rightarrow \quad \overrightarrow{AB}=[0-1,1-2,3-(-2)]=[-1,-1,5]$

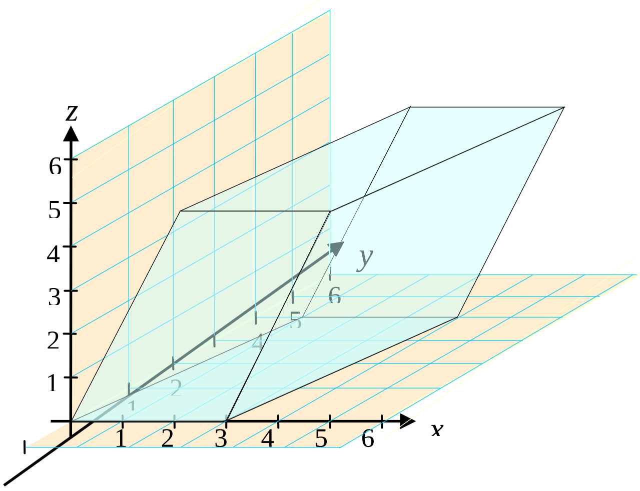
$AB=|\overrightarrow{AB}|=\sqrt{(-1)^2+(-1)^2+5^2}=\sqrt{27}=\sqrt{9\cdot 3}=\underline{\underline{3\sqrt{3}}}$

b) $A(3,0,3)$ og $B(-1,1,1) \quad \Rightarrow \quad \overrightarrow{AB}=[-1-3,1-0,1-3]=[-4,1,-2]$

$AB=|\overrightarrow{AB}|=\sqrt{(-4)^2+1^2+(-2)^2}=\underline{\underline{\sqrt{21}}}$

Oppgave 4.42

a)



b) $\overrightarrow{AB}=[3,0,0] \qquad \overrightarrow{AD}=[1,4,0] \qquad \overrightarrow{AE}=[1,1,4]$

$\overrightarrow{OC}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AB}+\overrightarrow{AD}=[3,0,0]+[1,4,0]=[4,4,0] \qquad \underline{\underline{C(4,4,0)}}$

$\overrightarrow{OF}=\overrightarrow{AB}+\overrightarrow{BF}=\overrightarrow{AB}+\overrightarrow{AE}=[3,0,0]+[1,1,4]=[4,1,4] \qquad \underline{\underline{F(4,1,4)}}$

$\overrightarrow{OG}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CG}=\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{AE}=[3,0,0]+[1,4,0]+[1,1,4]=[5,5,4] \qquad \underline{\underline{G(5,5,4)}}$

$\overrightarrow{OH}=\overrightarrow{AD}+\overrightarrow{BH}=\overrightarrow{AD}+\overrightarrow{AE}=[1,4,0]+[1,1,4]=[4,1,4] \qquad \underline{\underline{H(2,5,4)}}$

c) $AB = |\overrightarrow{AB}| = \sqrt{3^2 + 0^2 + 0^2} = \underline{\underline{3}}$

$AD = |\overrightarrow{AD}| = \sqrt{1^2 + 4^2 + 0^2} = \underline{\underline{\sqrt{17}}}$

$AE = |\overrightarrow{AE}| = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18} = \underline{\underline{3\sqrt{2}}}$

d) Høyden i parallellpipedet er lik z-koordinaten til E, siden grunnflata $ABCD$ ligger i xy-planet. Grunnflata er et parallellogram med grunnlinje lik førstekoordinaten til B (3) og høyde lik andrekoordinaten til D (4).

$V = G \cdot h = 3 \cdot 4 \cdot 4 = \underline{\underline{48}}$

Oppgave 4.43

a) Gitt punktene $A(1,1,0)$, $B(4,2,1)$, $D(2,4,2)$, $E(1,2,2)$

$\overrightarrow{AB} = [3,1,1] \qquad \overrightarrow{AD} = [1,3,2] \qquad \overrightarrow{AE} = [0,1,2]$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{AD} = [1,1,0] + [3,1,1] + [1,3,2] = [4,4,3] \qquad \underline{\underline{C(5,5,3)}}$

$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BF} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{AE} = [1,1,0] + [3,1,1] + [0,1,2] = [4,3,3] \qquad \underline{\underline{F(4,3,3)}}$

$\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CG} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AE}$

$= [1,1,0] + [3,1,1] + [1,3,2] + [0,1,2] = [5,6,5] \qquad \underline{\underline{G(5,6,5)}}$

$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{AD} + \overrightarrow{BH} = \overrightarrow{OA} + \overrightarrow{AD} + \overrightarrow{AE} = [1,1,0] + [1,3,2] + [0,1,2] = [2,5,4] \qquad \underline{\underline{H(2,5,4)}}$

b) $AB = |\overrightarrow{AB}| = \sqrt{3^2 + 1^2 + 1^2} = \underline{\underline{\sqrt{11}}}$

$AD = |\overrightarrow{AD}| = \sqrt{1^2 + 3^2 + 2^2} = \underline{\underline{\sqrt{14}}}$

$AE = |\overrightarrow{AE}| = \sqrt{0^2 + 1^2 + 2^2} = \underline{\underline{\sqrt{5}}}$

4.5 Skalarproduktet

Oppgave 4.50

a) $[1,2,-1]\cdot[2,3,3]=1\cdot 2+2\cdot 3+(-1)\cdot 3=2+6-3=\underline{\underline{5}}$

b) $[2,1,-1]\cdot[-3,2,1]=2\cdot(-3)+1\cdot 2+(-1)\cdot 1=-6+2-1=\underline{\underline{-5}}$

c) $[2,3,-4]\cdot[2,-4,-2]=2\cdot 2+3\cdot(-4)+(-4)\cdot(-2)=4-12+8=\underline{\underline{0}}$

Oppgave 4.51

a) $[1,-1,2]\perp[x,1,3] \quad\Leftrightarrow\quad [1,-1,2]\cdot[x,1,3]=0 \quad\Leftrightarrow\quad 1\cdot x+(-1)\cdot 1+2\cdot 3=0 \quad\Leftrightarrow$

$x-1+6=0 \quad\Leftrightarrow\quad \underline{\underline{x=-5}}$

b) $[3-x,2+x,1-2x]\perp[1,2,3] \quad\Leftrightarrow\quad [3-x,2+x,1-2x]\cdot[1,2,3]=0 \quad\Leftrightarrow$

$(3-x)\cdot 1+(2+x)\cdot 2+(1-2x)\cdot 3=0 \quad\Leftrightarrow\quad 3-x+4+2x+3-6x=0 \quad\Leftrightarrow$

$-5x=-10 \quad\Leftrightarrow\quad \underline{\underline{x=2}}$

Oppgave 4.52

a) Punktet D ligger på ei rett linje gjennom C som er parallell med linja gjennom A og B

$\Leftrightarrow\quad \overrightarrow{CD}\parallel\overrightarrow{AB} \quad\Leftrightarrow\quad \overrightarrow{CD}=t\cdot\overrightarrow{AB}$

$\overrightarrow{AD}=\overrightarrow{AC}+\overrightarrow{CD} \quad\Leftrightarrow\quad \underline{\underline{\overrightarrow{AD}=\overrightarrow{AC}+t\cdot\overrightarrow{AB}}}$

b) $\overrightarrow{AB}=[3-2,1-0,1-2]=[1,1,-1]$

$\overrightarrow{AC}=[1-2,2-0,4-2]=[-1,2,2]$

$\overrightarrow{AD}=[-1,2,2]+t\cdot[1,1,-1]=[-1+t,2+t,2-t]$

$\overrightarrow{AD}\perp\overrightarrow{AB} \quad\Leftrightarrow\quad \overrightarrow{AD}\cdot\overrightarrow{AB}=0$

$[-1+t,2+t,2-t]\cdot[1,1,-1]=0 \quad\Leftrightarrow$

$(-1+t)\cdot 1+(2+t)\cdot 1+(2-t)\cdot(-1)=0 \quad\Leftrightarrow\quad -1+t+2+t-2+t=0 \quad\Leftrightarrow$

$3t-1=0 \quad\Leftrightarrow\quad \underline{\underline{t=\frac{1}{3}}}$

c) $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}=[2,0,2]+\left[-1+\frac{1}{3},2+\frac{1}{3},2-\frac{1}{3}\right]=\left[2-\frac{2}{3},0+\frac{7}{3},2+\frac{5}{3}\right]=\left[\frac{4}{3},\frac{7}{3},\frac{11}{3}\right]$

$\underline{\underline{D\left(\frac{4}{3},\frac{7}{3},\frac{11}{3}\right)}}$

Oppgave 4.53

a) $$\cos u = \frac{[2,3,1]\cdot[3,-1,-2]}{|[2,3,1]|\cdot|[3,-1,-2]|} = \frac{2\cdot 3+3\cdot(-1)+1\cdot(-2)}{\sqrt{2^2+3^2+1^2}\cdot\sqrt{3^2+(-1)^2+(-2)^2}} = \frac{6-3-2}{\sqrt{14}\cdot\sqrt{14}} = \frac{1}{14}$$

$$\Rightarrow \quad \underline{\underline{u \approx 85,9°}}$$

b) $$\cos u = \frac{[2,-2,1]\cdot[-3,4,12]}{|[2,-2,1]|\cdot|[-3,4,12]|} = \frac{2\cdot(-3)+(-2)\cdot 4+1\cdot 12}{\sqrt{2^2+(-2)^2+1^2}\cdot\sqrt{(-3)^2+4^2+12^2}} = \frac{-6-8+12}{\sqrt{9}\cdot\sqrt{169}} = \frac{-2}{39}$$

$$\Rightarrow \quad \underline{\underline{u \approx 92,9°}}$$

c) $$\cos u = \frac{[1,3,-2]\cdot[2,2,4]}{|[1,3,-2]|\cdot|[2,2,4]|} = \frac{1\cdot 2+3\cdot 2+(-2)\cdot 4}{\sqrt{1^2+3^2+(-2)^2}\cdot\sqrt{2^2+2^2+4^2}} = \frac{2+6-8}{\sqrt{14}\cdot\sqrt{24}} = \frac{0}{\sqrt{14}\cdot\sqrt{24}} = 0$$

$$\Rightarrow \quad \underline{\underline{u = 90°}}$$

Oppgave 4.54

a) Hjørnene i trekant ABC: $A(1,0,1)$, $B(2,5,3)$ og $C(3,4,4)$

$$\overrightarrow{AB} = [2-1, 5-0, 3-1] = [1,5,2] \quad \Rightarrow \quad |\overrightarrow{AB}| = \sqrt{1^2+5^2+2^2} = \underline{\underline{\sqrt{30}}}$$

$$\overrightarrow{AC} = [3-1, 4-0, 4-1] = [2,4,3] \quad \Rightarrow \quad |\overrightarrow{AC}| = \sqrt{2^2+4^2+3^2} = \underline{\underline{\sqrt{29}}}$$

$$\overrightarrow{BC} = [3-2, 4-5, 4-3] = [1,-1,1] \quad \Rightarrow \quad |\overrightarrow{BC}| = \sqrt{1^2+(-1)^2+1^2} = \underline{\underline{\sqrt{3}}}$$

b) $$\cos\angle A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\cdot|\overrightarrow{AC}|} = \frac{[1,5,2]\cdot[2,4,3]}{\sqrt{30}\cdot\sqrt{29}} = \frac{28}{\sqrt{30}\cdot\sqrt{29}} \quad \Rightarrow \quad \underline{\underline{\angle A \approx 18,3°}}$$

$$\cos\angle B = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}|\cdot|\overrightarrow{BC}|} = \frac{[-1,-5,-2]\cdot[1,-1,1]}{\sqrt{30}\cdot\sqrt{3}} = \frac{2}{\sqrt{30}\cdot\sqrt{3}} \quad \Rightarrow \quad \underline{\underline{\angle B \approx 77,8°}}$$

$$\angle C = 180° - 18,3° - 77,8° = \underline{\underline{83,9°}}$$

c) $N(x, y, z) \quad \Rightarrow \quad \overrightarrow{AN} = [x-1, y, z-1] \qquad \overrightarrow{BN} = [x-2, y-5, z-3]$

N ligger på linja gjennom B og C $\Leftrightarrow \quad \overrightarrow{BN} \parallel \overrightarrow{BC} \quad \Leftrightarrow \quad \overrightarrow{BN} = t \cdot \overrightarrow{BC}$

$\Rightarrow \quad [x-2, y-5, z-3] = t \cdot [1, -1, 1] \quad \Leftrightarrow \quad x-2 = t \quad \wedge \quad y-5 = -t \quad \wedge \quad z-3 = t$

$\Leftrightarrow \quad x-2 = z-3 \quad \wedge \quad 5-y = z-3 \quad \Leftrightarrow \quad x = z-1 \quad \wedge \quad y = 8-z$

$AN \perp BC \quad \Leftrightarrow \quad \overrightarrow{AN} \cdot \overrightarrow{BC} = 0$

$\Rightarrow \quad [x-1, y, z-1] \cdot [1, -1, 1] = 0 \quad \Leftrightarrow \quad (x-1) \cdot 1 + y \cdot (-1) + (z-1) \cdot 1 = 0 \quad \Leftrightarrow$

$x - 1 - y + z - 1 = 0 \quad \Leftrightarrow \quad x - y + z = 2 \quad \Leftrightarrow \quad (z-1) - (8-z) + z = 2 \quad \Leftrightarrow$

$z - 1 - 8 + z + z = 2 \quad \Leftrightarrow \quad 3z = 11 \quad \Leftrightarrow$

$$\left.\begin{array}{l} z = \frac{11}{3} \\ x = z - 1 \quad \Rightarrow \quad x = \frac{11}{3} - 1 = \frac{8}{3} \\ y = 8 - z \quad \Rightarrow \quad y = 8 - \frac{11}{3} = \frac{13}{3} \end{array}\right\} \qquad \underline{\underline{N\left(\frac{8}{3}, \frac{13}{3}, \frac{11}{3}\right)}}$$

d) Med BC som grunnlinje blir AN høyde i trekant ABC.

$\overrightarrow{AN} = \left[\frac{8}{3} - 1, \frac{13}{3} - 0, \frac{11}{3} - 1\right] = \left[\frac{5}{3}, \frac{13}{3}, \frac{8}{3}\right] \quad \Rightarrow$

$$h = \left|\overrightarrow{AN}\right| = \sqrt{\left(\frac{5}{3}\right)^2 + \left(\frac{13}{3}\right)^2 + \left(\frac{8}{3}\right)^2} = \sqrt{\frac{25+169+64}{9}} = \sqrt{\frac{258}{9}} = \frac{\sqrt{258}}{3}$$

$$A_{\Delta ABC} = \frac{\left|\overrightarrow{BC}\right| \cdot h}{2} = \frac{\sqrt{3} \cdot \frac{\sqrt{258}}{3}}{2} = \frac{\sqrt{3} \cdot \sqrt{3 \cdot 86}}{2 \cdot 3} = \frac{\sqrt{3} \cdot \sqrt{3} \cdot \sqrt{86}}{6} = \underline{\underline{\frac{\sqrt{86}}{2}}}$$

e) Avstanden fra C til linja AB svarer til høyden på grunnlinja AB

$$\frac{\left|\overrightarrow{AB}\right| \cdot h}{2} = \frac{\sqrt{86}}{2} \quad \Leftrightarrow \quad h = \frac{\sqrt{86}}{\sqrt{30}} = \frac{\cancel{\sqrt{2}} \cdot \sqrt{43}}{\cancel{\sqrt{2}} \cdot \sqrt{15}} = \frac{\sqrt{43} \cdot \sqrt{15}}{\sqrt{15} \cdot \sqrt{15}} = \underline{\underline{\frac{\sqrt{645}}{15}}}$$

4.6 Regneregler for skalarproduktet

Oppgave 4.60

a) Gitt $|\vec{a}|=3$, $|\vec{b}|=4$, $|\vec{c}|=6$, $\angle(\vec{a},\vec{b})=90°$, $\angle(\vec{a},\vec{c})=45°$, $\angle(\vec{b},\vec{c})=60°$, $\vec{u}=\vec{a}+\vec{b}$ og $\vec{v}=2\vec{b}+\vec{c}$

$$\vec{a}\cdot\vec{b}=3\cdot 4\cdot\cos 90°=\underline{\underline{0}}$$

$$\vec{a}\cdot\vec{c}=3\cdot 6\cdot\cos 45°=18\cdot\frac{\sqrt{2}}{2}=\underline{\underline{9\sqrt{2}}}$$

$$\vec{b}\cdot\vec{c}=4\cdot 6\cdot\cos 60°=\underline{\underline{12}}$$

$$\vec{a}^{\,2}=3\cdot 3\cdot\cos 0°=3^2=\underline{\underline{9}}$$

$$\vec{b}^{\,2}=4^2=\underline{\underline{16}}$$

$$\vec{c}^{\,2}=6^2=\underline{\underline{36}}$$

b) $$\vec{u}\cdot\vec{v}=(\vec{a}+\vec{b})\cdot(2\vec{b}+\vec{c})=2\vec{a}\vec{b}+\vec{a}\vec{c}+2\vec{b}^{\,2}+\vec{b}\vec{c}=2\cdot 0+9\sqrt{2}+2\cdot 16+12=\underline{\underline{44+9\sqrt{2}}}$$

c) $$\vec{u}^{\,2}=(\vec{a}+\vec{b})^2=\vec{a}^{\,2}+2\vec{a}\vec{b}+\vec{b}^{\,2}=9+2\cdot 0+16=25$$

$$\vec{v}^{\,2}=(2\vec{b}+\vec{c})^2=4\vec{b}^{\,2}+4\vec{b}\vec{c}+\vec{c}^{\,2}=4\cdot 16+4\cdot 4\cdot 6\cdot\frac{1}{2}+36=64+48+36=148$$

$$|\vec{u}|=\sqrt{\vec{u}^{\,2}}=\sqrt{25}=\underline{\underline{5}}$$

$$|\vec{v}|=\sqrt{\vec{v}^{\,2}}=\underline{\underline{\sqrt{148}}}$$

$$\cos\angle(\vec{u},\vec{v})=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}=\frac{44+9\sqrt{2}}{5\cdot\sqrt{148}}\quad\Rightarrow\quad\underline{\underline{\angle(\vec{u},\vec{v})\approx 21{,}2°}}$$

Oppgave 4.61

a) Gitt $|\vec{a}|=2$, $|\vec{b}|=3$, $|\vec{c}|=5$, $\angle(\vec{a},\vec{b})=60°$, $\angle(\vec{a},\vec{c})=90°$, $\angle(\vec{b},\vec{c})=90°$,
$\vec{u}=\vec{a}+\vec{b}+\vec{c}$ og $\vec{v}=2\vec{a}+\vec{b}-\vec{c}$

$$\vec{a}\cdot\vec{b}=2\cdot 3\cdot\cos 60°=\underline{\underline{3}}$$

$$\vec{a}\cdot\vec{c}=2\cdot 5\cdot\cos 90°=\underline{\underline{0}}$$

$$\vec{b}\cdot\vec{c}=3\cdot 5\cdot\cos 90°=\underline{\underline{0}}$$

$$\vec{a}^{\,2}=2^2=\underline{\underline{4}}$$

$$\vec{b}^{\,2}=3^2=\underline{\underline{9}}$$

$$\vec{c}^{\,2}=5^2=\underline{\underline{25}}$$

b) $\vec{u}\cdot\vec{v}=\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\left(2\vec{a}+\vec{b}-\vec{c}\right)=2\vec{a}^{\,2}+\vec{a}\vec{b}-\vec{a}\vec{c}+2\vec{a}\vec{b}+\vec{b}^{\,2}-\vec{b}\vec{c}+2\vec{a}\vec{c}+\vec{b}\vec{c}-\vec{c}^{\,2}$

$=2\vec{a}^{\,2}+3\vec{a}\vec{b}+\vec{a}\vec{c}+\vec{b}^{\,2}-\vec{c}^{\,2}=2\cdot4+3\cdot3+0+9-25=\underline{\underline{1}}$

c) $\vec{u}^{\,2}=\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)=\vec{a}^{\,2}+\vec{a}\vec{b}+\vec{a}\vec{c}+\vec{a}\vec{b}+\vec{b}^{\,2}+\vec{b}\vec{c}+\vec{a}\vec{c}+\vec{b}\vec{c}+\vec{c}^{\,2}$

$=\vec{a}^{\,2}+2\vec{a}\vec{b}+2\vec{a}\vec{c}+2\vec{b}\vec{c}+\vec{b}^{\,2}+\vec{c}^{\,2}=4+2\cdot3+2\cdot0+2\cdot0+9+25=44$

$\vec{v}^{\,2}=\left(2\vec{a}+\vec{b}-\vec{c}\right)\cdot\left(2\vec{a}+\vec{b}-\vec{c}\right)=4\vec{a}^{\,2}+2\vec{a}\vec{b}-2\vec{a}\vec{c}+2\vec{a}\vec{b}+\vec{b}^{\,2}-\vec{b}\vec{c}-2\vec{a}\vec{c}-\vec{b}\vec{c}+\vec{c}^{\,2}$

$=4\vec{a}^{\,2}+4\vec{a}\vec{b}-4\vec{a}\vec{c}-2\vec{b}\vec{c}+\vec{b}^{\,2}+\vec{c}^{\,2}=4\cdot4+4\cdot3-4\cdot0-2\cdot0+9+25=62$

$\left|\vec{u}\right|=\sqrt{\vec{u}^{\,2}}=\underline{\underline{\sqrt{44}}}$

$\left|\vec{v}\right|=\sqrt{\vec{v}^{\,2}}=\underline{\underline{\sqrt{62}}}$

$\cos\angle\left(\vec{u},\vec{v}\right)=\dfrac{\vec{u}\cdot\vec{v}}{\left|\vec{u}\right|\cdot\left|\vec{v}\right|}=\dfrac{1}{\sqrt{44}\cdot\sqrt{62}}\quad\Rightarrow\quad\underline{\underline{\angle\left(\vec{u},\vec{v}\right)\approx 88{,}9^\circ}}$

Oppgave 4.62

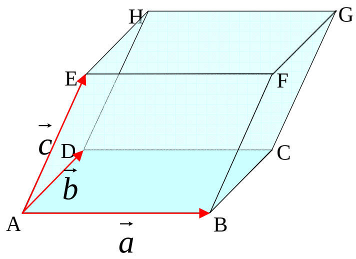
$\left(\vec{a}+\vec{b}\right)^2=\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right)=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}=\underline{\underline{\vec{a}^{\,2}+2\vec{a}\cdot\vec{b}+\vec{b}^{\,2}}}$

$\left(\vec{a}-\vec{b}\right)^2=\left(\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}=\underline{\underline{\vec{a}^{\,2}-2\vec{a}\cdot\vec{b}+\vec{b}^{\,2}}}$

$\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{b}=\underline{\underline{\vec{a}^{\,2}-\vec{b}^{\,2}}}$

Oppgave 4.63

a)



Innfører $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{AD}$ og $\vec{c} = \overrightarrow{AE}$

$|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5$, $\angle(\vec{a},\vec{b}) = 90°$, $\angle(\vec{a},\vec{c}) = 60°$, $\angle(\vec{b},\vec{c}) = 60°$

$\vec{a}\cdot\vec{b} = 3\cdot 4\cdot\cos 90° = 0 \qquad \vec{a}\cdot\vec{c} = 3\cdot 5\cdot\cos 60° = \frac{15}{2}$

$\vec{b}\cdot\vec{c} = 4\cdot 5\cdot\cos 60° = 10 \qquad \vec{a}^2 = 9 \qquad \vec{b}^2 = 16 \qquad \vec{c}^2 = 25$

$$\overrightarrow{AC} = \vec{a}+\vec{b} \;\Rightarrow\; \left|\overrightarrow{AC}\right|^2 = \left(\vec{a}+\vec{b}\right)^2 = \vec{a}^2 + 2\vec{a}\cdot\vec{b} + \vec{b}^2 = 9 + 2\cdot 0 + 16 = 25$$

$$AC = \left|\overrightarrow{AC}\right| = \sqrt{25} = \underline{\underline{5}}$$

$$\cos\angle BAC = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AC}\right|} = \frac{\vec{a}\cdot\left(\vec{a}+\vec{b}\right)}{3\cdot 5} = \frac{\vec{a}^2 + \vec{a}\cdot\vec{b}}{15} = \frac{9+0}{15} = \frac{9}{15}$$

$$\Rightarrow \underline{\underline{\angle BAC \approx 53{,}1°}}$$

b)

$$\overrightarrow{AG} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CG} = \vec{a} + \vec{b} + \vec{c}$$

$$\left|\overrightarrow{AG}\right|^2 = \left(\vec{a}+\vec{b}+\vec{c}\right)^2 = \vec{a}^2 + 2\vec{a}\vec{b} + 2\vec{a}\vec{c} + 2\vec{b}\vec{c} + \vec{b}^2 + \vec{c}^2 = 9 + 2\cdot 0 + 2\cdot\frac{15}{2} + 2\cdot 10 + 16 + 25 = 85$$

$$AG = \left|\overrightarrow{AG}\right| = \underline{\underline{\sqrt{85}}}$$

c)

$$\cos\angle CAG = \frac{\overrightarrow{AC}\cdot\overrightarrow{AG}}{\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{AG}\right|} = \frac{\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)}{5\cdot\sqrt{103}} = \frac{\vec{a}^2 + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{b} + \vec{b}^2 + \vec{b}\cdot\vec{c}}{5\cdot\sqrt{103}}$$

$$= \frac{9 + 0 + \frac{15}{2} + 0 + 16 + 10}{5\cdot\sqrt{85}} = \frac{\frac{85}{2}}{5\cdot\sqrt{85}} = \frac{85\cdot\sqrt{85}\cdot\sqrt{85}}{10\cdot\sqrt{85}\cdot\sqrt{85}} = \frac{\sqrt{85}}{10}$$

$$\Rightarrow \underline{\underline{\angle CAG \approx 22{,}8°}}$$

d)

M er midtpunktet på AG $\;\Leftrightarrow\;$ $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AG} = \frac{1}{2}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)$

M ligger på linja CE $\;\Leftrightarrow\;$ $\overrightarrow{CM} = t\cdot\overrightarrow{CE}$

$$\overrightarrow{CE} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AE} = -\vec{b} - \vec{a} + \vec{c} \quad\Rightarrow\quad \overrightarrow{CM} = t\cdot\left(-\vec{a} - \vec{b} + \vec{c}\right)$$

$$\overrightarrow{CM} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AM} = -\vec{b} - \vec{a} + \frac{1}{2}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right) = -\vec{b} - \vec{a} + \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c} = -\frac{1}{2}\vec{a} - \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}$$

$$t\cdot\left(-\vec{a} - \vec{b} + \vec{c}\right) = -\frac{1}{2}\vec{a} - \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c} \quad\Leftrightarrow\quad t = \frac{1}{2} \quad\Rightarrow\quad \overrightarrow{CM} = \frac{1}{2}\cdot\overrightarrow{CE}$$

<u>Midtpunktet på AG ligger på linja CE.</u>

e) $CM \perp AG \quad \Leftrightarrow \quad \overrightarrow{CM}\cdot\overrightarrow{AG}=0$

$\overrightarrow{CM}=\frac{1}{2}\left(-\vec{a}-\vec{b}+\vec{c}\right)$

$$\overrightarrow{CM}\cdot\overrightarrow{AG}=\left(-\tfrac{1}{2}\vec{a}-\tfrac{1}{2}\vec{b}+\tfrac{1}{2}\vec{c}\right)\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)$$
$$=-\tfrac{1}{2}\vec{a}^{\,2}-\tfrac{1}{2}\vec{a}\cdot\vec{b}-\tfrac{1}{2}\vec{a}\cdot\vec{c}-\tfrac{1}{2}\vec{a}\cdot\vec{b}-\tfrac{1}{2}\vec{b}^{\,2}-\tfrac{1}{2}\vec{b}\cdot\vec{c}+\tfrac{1}{2}\vec{a}\cdot\vec{c}+\tfrac{1}{2}\vec{b}\cdot\vec{c}+\tfrac{1}{2}\vec{c}^{\,2}$$
$$=-\tfrac{1}{2}\vec{a}^{\,2}-\vec{a}\cdot\vec{b}-\tfrac{1}{2}\vec{b}^{\,2}+\tfrac{1}{2}\vec{c}^{\,2}=-\tfrac{1}{2}\cdot 9-0-\tfrac{1}{2}\cdot 16+\tfrac{1}{2}\cdot 25=0$$

$\underline{\underline{CM \perp AG}}$

Oppgave 4.64

a) Innfører $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{AD}$ og $\vec{c}=\overrightarrow{AE}$

$|\vec{a}|=4$, $|\vec{b}|=4$, $|\vec{c}|=4$, $\angle(\vec{a},\vec{b})=60°$, $\angle(\vec{a},\vec{c})=90°$, $\angle(\vec{b},\vec{c})=90°$

$\vec{a}\cdot\vec{b}=4\cdot 4\cdot\cos 60°=8 \qquad \vec{a}\cdot\vec{c}=4\cdot 4\cdot\cos 90°=0$

$\vec{b}\cdot\vec{c}=4\cdot 4\cdot\cos 90°=0 \qquad \vec{a}^{\,2}=16 \qquad \vec{b}^{\,2}=16 \qquad \vec{c}^{\,2}=16$

$$\overrightarrow{AG}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CG}=\vec{a}+\vec{b}+\vec{c}$$
$$\left|\overrightarrow{AG}\right|^2=\left(\vec{a}+\vec{b}+\vec{c}\right)^2=\vec{a}^{\,2}+2\vec{a}\vec{b}+2\vec{a}\vec{c}+2\vec{b}\vec{c}+\vec{b}^{\,2}+\vec{c}^{\,2}=16+2\cdot 8+2\cdot 0+2\cdot 0+16+16=64$$
$$AG=\left|\overrightarrow{AG}\right|=\sqrt{64}=\underline{\underline{8}}$$

$$\overrightarrow{CE}=\overrightarrow{CB}+\overrightarrow{BA}+\overrightarrow{AE}=-\vec{b}-\vec{a}+\vec{c}$$
$$\left|\overrightarrow{CE}\right|^2=\left(\vec{c}-\vec{a}-\vec{b}\right)^2=\vec{c}^{\,2}-2\vec{c}\vec{a}-2\vec{c}\vec{b}+2\vec{b}\vec{a}+\vec{a}^{\,2}+\vec{b}^{\,2}=16-2\cdot 0-2\cdot 0+2\cdot 8+16+16=64$$
$$CE=\left|\overrightarrow{CE}\right|=\sqrt{64}=\underline{\underline{8}}$$

b) $\left|\overrightarrow{AC}\right|^2=\left(\vec{a}+\vec{b}\right)^2=\vec{a}^{\,2}+2\vec{a}\vec{b}+\vec{b}^{\,2}=16+2\cdot 8+16=48 \quad\Rightarrow\quad \left|\overrightarrow{AC}\right|=\sqrt{48}$

$$\cos\angle ACE=\frac{\overrightarrow{CA}\cdot\overrightarrow{CE}}{\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{CE}\right|}=\frac{\left(-\vec{b}-\vec{a}\right)\cdot\left(-\vec{a}-\vec{b}+\vec{c}\right)}{\sqrt{48}\cdot 8}=\frac{\vec{a}\cdot\vec{b}+\vec{b}^{\,2}-\vec{b}\cdot\vec{c}+\vec{a}^{\,2}+\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}}{\sqrt{48}\cdot 8}$$
$$=\frac{8+16-0+16+8-0}{\sqrt{48}\cdot 8}=\frac{48}{\sqrt{48}\cdot 8}$$

$\Rightarrow \quad \underline{\underline{\angle ACE=30°}}$

c) M er midtpunktet på $AG \quad \Leftrightarrow \quad \overrightarrow{AM} = \frac{1}{2}\overrightarrow{AG} = \frac{1}{2}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)$

M ligger på linja $CE \quad \Leftrightarrow \quad \overrightarrow{CM} = t\cdot\overrightarrow{CE}$

$\overrightarrow{CE} = \overrightarrow{CB}+\overrightarrow{BA}+\overrightarrow{AE} = -\vec{b}-\vec{a}+\vec{c} \quad \Rightarrow \quad \overrightarrow{CM} = t\cdot\left(-\vec{a}-\vec{b}+\vec{c}\right)$

$\overrightarrow{CM} = \overrightarrow{CB}+\overrightarrow{BA}+\overrightarrow{AM} = -\vec{b}-\vec{a}+\frac{1}{2}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right) = -\vec{b}-\vec{a}+\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c} = -\frac{1}{2}\vec{a}-\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c}$

$t\cdot\left(-\vec{a}-\vec{b}+\vec{c}\right) = -\frac{1}{2}\vec{a}-\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c} \quad \Leftrightarrow \quad t=\frac{1}{2} \quad \Rightarrow \quad \overrightarrow{CM} = \frac{1}{2}\cdot\overrightarrow{CE}$

Midtpunktet på AG ligger på linja CE.

d) $\overrightarrow{MA} = -\frac{1}{2}\overrightarrow{AG} = -\frac{1}{2}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right) \quad \Rightarrow \quad \left|\overrightarrow{MA}\right| = \frac{1}{2}\left|\overrightarrow{AG}\right| = \frac{1}{2}\cdot 8 = 4$

$\overrightarrow{ME} = \overrightarrow{MA}+\overrightarrow{AE} = -\frac{1}{2}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)+\vec{c} = -\frac{1}{2}\cdot\left(\vec{a}+\vec{b}-\vec{c}\right)$

$\left|\overrightarrow{ME}\right|^2 = \left[-\frac{1}{2}\cdot\left(\vec{a}+\vec{b}-\vec{c}\right)\right]^2 = \frac{1}{4}\left(\vec{a}^2+2\vec{a}\vec{b}-2\vec{a}\vec{c}-2\vec{b}\vec{c}+\vec{b}^2+\vec{c}^2\right)$

$= \frac{1}{4}\left(16+2\cdot 8-2\cdot 0-2\cdot 0+16+16\right) = 16$

$\left|\overrightarrow{ME}\right| = \sqrt{16} = 4$

$\overrightarrow{MA}\cdot\overrightarrow{ME} = -\frac{1}{2}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\left(-\frac{1}{2}\right)\cdot\left(\vec{a}+\vec{b}-\vec{c}\right) = \frac{1}{4}\left(\vec{a}^2+\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}+\vec{a}\cdot\vec{b}+\vec{b}^2-\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}-\vec{c}^2\right)$

$= \frac{1}{4}\left(\vec{a}^2+2\vec{a}\cdot\vec{b}+\vec{b}^2-\vec{c}^2\right) = \frac{1}{4}\left(16+2\cdot 8+16-16\right) = 8$

$\cos\angle AME = \dfrac{\overrightarrow{MA}\cdot\overrightarrow{ME}}{\left|\overrightarrow{MA}\right|\cdot\left|\overrightarrow{ME}\right|} = \dfrac{8}{4\cdot 4} = \dfrac{8}{16} = \dfrac{1}{2} \quad \Rightarrow \quad \angle AME = 60°$

4.7 Determinanter

Oppgave 4.70

a) $\begin{vmatrix} 7 & 3 \\ 2 & 1 \end{vmatrix} = 7 \cdot 1 - 3 \cdot 2 = 7 - 6 = \underline{\underline{1}}$

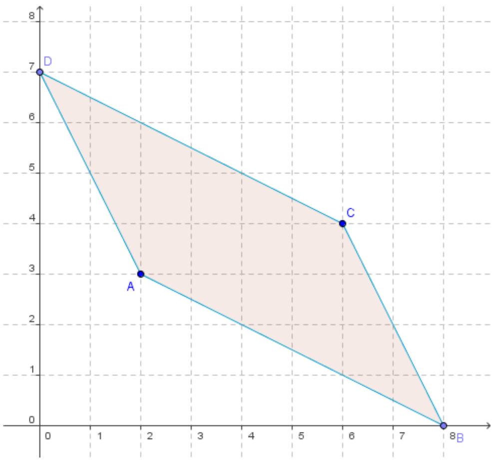
b) $\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = 3 \cdot 2 - (-2) \cdot 1 = 6 + 2 = \underline{\underline{8}}$

c) $\begin{vmatrix} 5 & 4 \\ -3 & 2 \end{vmatrix} = 5 \cdot 2 - 4 \cdot (-3) = 10 + 12 = \underline{\underline{22}}$

d) $\begin{vmatrix} 2x & x \\ 3 & 2 \end{vmatrix} = 2x \cdot 2 - x \cdot 3 = 4x - 3x = \underline{\underline{x}}$

e) $\begin{vmatrix} 4x & 2x \\ 2 & 1 \end{vmatrix} = 4x \cdot 1 - 2x \cdot 2 = 4x - 4x = \underline{\underline{0}}$

Oppgave 4.71



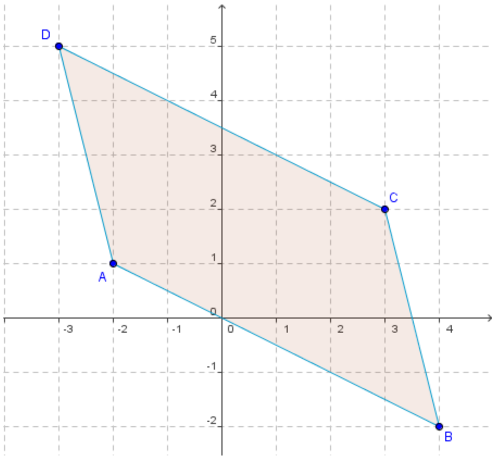
$\overrightarrow{AB} = [8-2, 0-3] = [6, -3]$

$\overrightarrow{AC} = [0-2, 7-3] = [-2, 4]$

$A = \begin{vmatrix} 6 & -3 \\ -2 & 4 \end{vmatrix} = 6 \cdot 4 - (-3) \cdot (-2) = 24 - 6 = \underline{\underline{18}}$

Oppgave 4.72

a)



$$\overrightarrow{AB} = \left[4-(-2), -2-1\right] = [6, -3]$$

$$\overrightarrow{BC} = \overrightarrow{AD} = \left[3-4, 2-(-2)\right] = [-1, 4]$$

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = [-2, 1] + [-1, 4] = [-3, 5] \quad \underline{\underline{D(-3, 5)}}$$

b)

$$A = \begin{vmatrix} 6 & -3 \\ -1 & 4 \end{vmatrix} = 6 \cdot 4 - (-3) \cdot (-1) = 24 - 3 = \underline{\underline{21}}$$

c)

$$\left|\overrightarrow{AB}\right| = \sqrt{6^2 + (-3)^2} = \sqrt{36+9} = \sqrt{45}$$

$$\left|\overrightarrow{AB}\right| \cdot h = 21 \quad \Leftrightarrow \quad h = \frac{21}{\sqrt{45}} = \frac{21}{\sqrt{9} \cdot \sqrt{5}} = \frac{21}{3\sqrt{5}} = \underline{\underline{\frac{7}{\sqrt{5}}}}$$

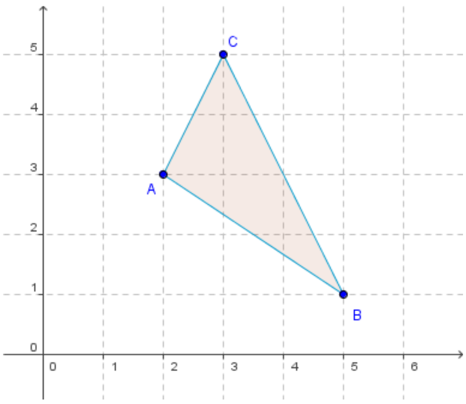
d)

$$\left|\overrightarrow{BC}\right| = \sqrt{(-1)^2 + 4^2} = \sqrt{1+16} = \sqrt{17}$$

$$\left|\overrightarrow{BC}\right| \cdot h = 21 \quad \Leftrightarrow \quad \underline{\underline{h = \frac{21}{\sqrt{17}}}}$$

Oppgave 4.73

a)



$$\overrightarrow{AB} = [5-2, 1-3] = [3, -2]$$

$$\overrightarrow{AC} = [3-2, 5-3] = [1, 2]$$

$$A = \frac{1}{2} \cdot \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = \tfrac{1}{2}\left(3 \cdot 2 - (-2) \cdot 1\right) = \tfrac{1}{2}(6+2) = \underline{\underline{4}}$$

b) $\overrightarrow{BC} = [3-5, 5-1] = [-2, 4]$ $\quad \left|\overrightarrow{BC}\right| = \sqrt{(-2)^2 + 4^2} = \sqrt{4+16} = \sqrt{20}$

$$\frac{\left|\overrightarrow{BC}\right| \cdot h}{2} = 4 \;\Leftrightarrow\; h = \frac{8}{\sqrt{20}} = \frac{8}{\sqrt{4}\cdot\sqrt{5}} = \frac{8}{2\sqrt{5}} = \underline{\underline{\frac{4}{\sqrt{5}}}}$$

c) $\left|\overrightarrow{AC}\right| = \sqrt{1^2 + 2^2} = \sqrt{5}$ $\quad \Rightarrow \quad \frac{\left|\overrightarrow{AC}\right| \cdot h}{2} = 4 \;\Leftrightarrow\; \underline{\underline{h = \frac{8}{\sqrt{5}}}}$

d) $\left|\overrightarrow{AB}\right| = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$ $\quad \Rightarrow \quad \frac{\left|\overrightarrow{AB}\right| \cdot h}{2} = 4 \;\Leftrightarrow\; \underline{\underline{h = \frac{8}{\sqrt{13}}}}$

Oppgave 4.74

a) To vektorer er parallelle

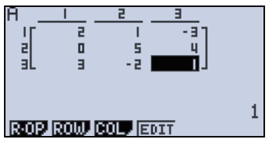
$\Leftrightarrow$ Arealet av det parallellogrammet som vektorene bestemmer, er 0

$\Leftrightarrow$ Determinanten til vektorene er 0

b) $\begin{vmatrix} 18 & -24 \\ -45 & 60 \end{vmatrix} = 18\cdot 60 - (-24)\cdot(-45) = 1080 - 1080 = 0 \quad \Rightarrow \quad \underline{\underline{[18, -24] \parallel [-45, 60]}}$

Oppgave 4.75

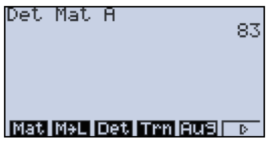
a) $\begin{vmatrix} 2 & 1 & -3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{vmatrix} = 2\cdot\begin{vmatrix} 5 & 4 \\ -2 & 1 \end{vmatrix} - 1\cdot\begin{vmatrix} 0 & 4 \\ 3 & 1 \end{vmatrix} + (-3)\cdot\begin{vmatrix} 0 & 5 \\ 3 & -2 \end{vmatrix}$



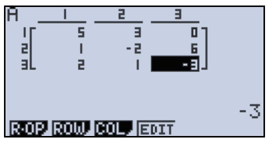
$= 2\cdot(5\cdot 1 - 4\cdot(-2)) - 1\cdot(0\cdot 1 - 4\cdot 3) - 3\cdot(0\cdot(-2) - 5\cdot 3)$

$= 2\cdot(5+8) - 1\cdot(0-12) - 3\cdot(0-15)$

$= 2\cdot 13 - 1\cdot(-12) - 3\cdot(-15) = 26 + 12 + 45 = \underline{\underline{83}}$

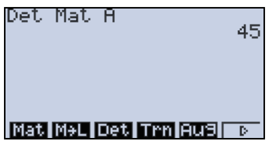


b) $\begin{vmatrix} 5 & 3 & 0 \\ 1 & -2 & 6 \\ 2 & 1 & -3 \end{vmatrix} = 5\cdot\begin{vmatrix} -2 & 6 \\ 1 & -3 \end{vmatrix} - 3\cdot\begin{vmatrix} 1 & 6 \\ 2 & -3 \end{vmatrix} + 0\cdot\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$

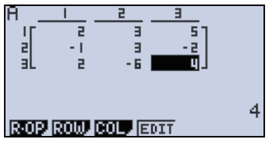


$= 5\cdot((-2)\cdot(-3) - 6\cdot 1) - 3\cdot(1\cdot(-3) - 6\cdot 2) - 0\cdot(1\cdot 1 - (-2)\cdot 2)$

$= 5\cdot(6-6) - 3\cdot(-3-12) - 0\cdot(1+4) = 5\cdot 0 - 3\cdot(-15) - 0\cdot 5 = \underline{\underline{45}}$



c) $\begin{vmatrix} 2 & 3 & 5 \\ -1 & 3 & -2 \\ 2 & -6 & 4 \end{vmatrix} = 2\cdot\begin{vmatrix} 3 & -2 \\ -6 & 4 \end{vmatrix} - 3\cdot\begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} + 5\cdot\begin{vmatrix} -1 & 3 \\ 2 & -6 \end{vmatrix}$



$$= 2\cdot\big(3\cdot 4-(-2)\cdot(-6)\big)-3\cdot\big((-1)\cdot 4-(-2)\cdot 2\big)-5\cdot\big((-1)\cdot(-6)-3\cdot 2\big)$$

$$= 2\cdot(12-12)-3\cdot(-4+4)-5\cdot(6-6) = 2\cdot 0-3\cdot 0-5\cdot 0 = \underline{\underline{0}}$$

4.8 Vektorproduktet

Oppgave 4.80

a)

$$[1,2,-1]\times[2,3,3]=\begin{vmatrix}\vec{e_1} & \vec{e_2} & \vec{e_3}\\ 1 & 2 & -1\\ 2 & 3 & 3\end{vmatrix}=\left[\begin{vmatrix}2 & -1\\ 3 & 3\end{vmatrix},-\begin{vmatrix}1 & -1\\ 2 & 3\end{vmatrix},\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}\right]$$

$$=\left[6-(-3),-(3-(-2)),3-4\right]=\underline{\underline{[9,-5,-1]}}$$

$$[1,2,-1]\cdot[9,-5,-1]=9-10+1=0 \quad\Rightarrow\quad \underline{\underline{[1,2,-1]\times[2,3,3]\perp[1,2,-1]}}$$

$$[2,3,3]\cdot[9,-5,-1]=18-15-3=0 \quad\Rightarrow\quad \underline{\underline{[1,2,-1]\times[2,3,3]\perp[2,3,3]}}$$

b)

$$[2,1,-1]\times[-3,2,1]=\begin{vmatrix}\vec{e_1} & \vec{e_2} & \vec{e_3}\\ 2 & 1 & -1\\ -3 & 2 & 1\end{vmatrix}=\left[\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix},-\begin{vmatrix}2 & -1\\ -3 & 1\end{vmatrix},\begin{vmatrix}2 & 1\\ -3 & 2\end{vmatrix}\right]$$

$$=\left[1-(-2),-(2-3),4-(-3)\right]=\underline{\underline{[3,1,7]}}$$

$$[2,1,-1]\cdot[3,1,7]=6+1-7=0 \quad\Rightarrow\quad \underline{\underline{[2,1,-1]\times[-3,2,1]\perp[2,1,-1]}}$$

$$[-3,2,1]\cdot[3,1,7]=-9+2+7=0 \quad\Rightarrow\quad \underline{\underline{[2,1,-1]\times[-3,2,1]\perp[-3,2,1]}}$$

c)

$$[2,3,-4]\times[2,-4,-2]=\begin{vmatrix}\vec{e_1} & \vec{e_2} & \vec{e_3}\\ 2 & 3 & -4\\ 2 & -4 & -2\end{vmatrix}=\left[\begin{vmatrix}3 & -4\\ -4 & -2\end{vmatrix},-\begin{vmatrix}2 & -4\\ 2 & -2\end{vmatrix},\begin{vmatrix}2 & 3\\ 2 & -4\end{vmatrix}\right]$$

$$=\left[-6-16,-((-4)-(-8)),(-8)-6\right]=\underline{\underline{[-22,-4,-14]}}$$

$$[2,3,-4]\cdot[-22,-4,-14]=-44-12+56=0 \quad\Rightarrow\quad \underline{\underline{[2,3,-4]\times[2,-4,-2]\perp[2,3,-4]}}$$

$$[2,-4,-2]\cdot[-22,-4,-14]=-44+16+28=0 \quad\Rightarrow\quad \underline{\underline{[2,3,-4]\times[2,-4,-2]\perp[2,-4,-2]}}$$

d)

$$[4,6,-2]\times[6,9,-3]=\begin{vmatrix}\vec{e_1} & \vec{e_2} & \vec{e_3}\\ 4 & 6 & -2\\ 6 & 9 & -3\end{vmatrix}=\left[\begin{vmatrix}6 & -2\\ 9 & -3\end{vmatrix},-\begin{vmatrix}4 & -2\\ 6 & -3\end{vmatrix},\begin{vmatrix}4 & 6\\ 6 & 9\end{vmatrix}\right]$$

$$=\left[-18-(-18),-((-12)-(-12)),36-36\right]=\underline{\underline{[0,0,0]}}$$

$$[4,6,-2]\cdot[0,0,0]=0 \quad\Rightarrow\quad \underline{\underline{[4,6,-2]\times[6,9,-3]\perp[4,6,-2]}}$$

$$[6,9,-3]\cdot[0,0,0]=0 \quad\Rightarrow\quad \underline{\underline{[4,6,-2]\times[6,9,-3]\perp[6,9,-3]}}$$

Oppgave 4.81

a) Gitt punktene $A(-1,1,1)$, $B(3,3,4)$ og $D(1,3,2)$

$\overrightarrow{AB}=[3-(-1),3-1,4-1]=[4,2,3]$

$\overrightarrow{AD}=[1-(-1),3-1,2-1]=[2,2,1]$

$C(x,y,z) \quad \Rightarrow \quad \overrightarrow{BC}=[x-3,y-3,z-4]$

$ABCD$ et parallellogram $\Leftrightarrow \quad \overrightarrow{BC}=\overrightarrow{AD}$

$[x-3,y-3,z-4]=[2,2,1] \quad \Leftrightarrow \quad x-3=2 \quad \wedge \quad y-3=2 \quad \wedge \quad z-4=1$

$\Leftrightarrow \quad x=5 \quad \wedge \quad y=5 \quad \wedge \quad z=5 \qquad \underline{\underline{C(5,5,5)}}$

b) $\overrightarrow{AB}\times\overrightarrow{AD}=[4,2,3]\times[2,2,1]=\left[\begin{vmatrix}2&3\\2&1\end{vmatrix},-\begin{vmatrix}4&3\\2&1\end{vmatrix},\begin{vmatrix}4&2\\2&2\end{vmatrix}\right]=[2-6,-(4-6),8-4]=[-4,2,4]$

$A=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|=\sqrt{(-4)^2+2^2+4^2}=\sqrt{16+4+16}=\sqrt{36}=\underline{\underline{6}}$

c) $\left|\overrightarrow{AB}\right|=\sqrt{4^2+2^2+3^2}=\sqrt{16+4+9}=\sqrt{29} \qquad A=\left|\overrightarrow{AB}\right|\cdot h \quad \Rightarrow \quad \underline{\underline{h=\frac{6}{\sqrt{29}}}}$

Oppgave 4.82

a) Gitt punktene $A(-2,1,-1)$, $B(2,3,3)$ og $C(1,5,5)$.

$\overrightarrow{AB}=[2-(-2),3-1,3-(-1)]=[4,2,4]$

$\overrightarrow{AC}=[1-(-2),5-1,5-(-1)]=[3,4,6]$

$\overrightarrow{AB}\times\overrightarrow{AC}=[4,2,4]\times[3,4,6]=\left[\begin{vmatrix}2&4\\4&6\end{vmatrix},-\begin{vmatrix}4&4\\3&6\end{vmatrix},\begin{vmatrix}4&2\\3&4\end{vmatrix}\right]=[12-16,-(24-12),16-6]=[-4,-12,10]$

$A=\frac{1}{2}\cdot\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\cdot\sqrt{(-4)^2+(-12)^2+10^2}=\frac{1}{2}\cdot\sqrt{16+144+100}=\frac{1}{2}\cdot\sqrt{260}=\frac{1}{2}\cdot\sqrt{4}\cdot\sqrt{65}=\underline{\underline{\sqrt{65}}}$

b) Avstanden fra C til linja gjennom A og B tilsvarer høyden på grunnlinja AB.

$\left|\overrightarrow{AB}\right|=\sqrt{4^2+2^2+4^2}=\sqrt{16+4+16}=\sqrt{36}=6$

$A=\frac{\left|\overrightarrow{AB}\right|\cdot h}{2} \quad \Rightarrow \quad \frac{6h}{2}=\sqrt{65} \quad \Leftrightarrow \quad \underline{\underline{h=\frac{\sqrt{65}}{3}}}$

Oppgave 4.83

Gitt punktene $A(1,1,3)$, $B(3,-1,1)$, $C(5,3,2)$ og $D(2,4,4)$

$\overrightarrow{AB} = [3-1,-1-1,1-3] = [2,-2,-2]$

$\overrightarrow{AD} = [2-1,4-1,4-3] = [1,3,1]$

$\overrightarrow{BC} = \left[5-3,3-(-1),2-1\right] = [2,4,1]$

$\overrightarrow{BD} = \left[2-3,4-(-1),4-1\right] = [-1,5,3]$

$$\overrightarrow{AB} \times \overrightarrow{AD} = [2,-2,-2] \times [1,3,1] = \left[\begin{vmatrix} -2 & -2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & -2 \\ 1 & 3 \end{vmatrix}\right]$$

$$= \left[-2-(-6), -\left(2-(-2)\right), 6-(-2)\right] = [4,-4,8]$$

$$\overrightarrow{BC} \times \overrightarrow{BD} = [2,4,1] \times [-1,5,3] = \left[\begin{vmatrix} 4 & 1 \\ 5 & 3 \end{vmatrix}, -\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}, \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix}\right]$$

$$= \left[12-5, -\left(6-(-1)\right), 10-(-4)\right] = [7,-7,14]$$

$$A_{\Delta ABD} = \tfrac{1}{2} \cdot \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \tfrac{1}{2} \cdot \sqrt{4^2 + (-4)^2 + 8^2} = \tfrac{1}{2} \cdot \sqrt{16+16+64} = \tfrac{1}{2} \cdot \sqrt{96} = \tfrac{1}{2} \cdot \sqrt{16} \cdot \sqrt{6} = 2\sqrt{6}$$

$$A_{\Delta BCD} = \tfrac{1}{2} \cdot \left|\overrightarrow{BC} \times \overrightarrow{BD}\right| = \tfrac{1}{2} \cdot \sqrt{7^2 + (-7)^2 + 14^2} = \tfrac{1}{2} \cdot \sqrt{49+49+196} = \tfrac{1}{2} \cdot \sqrt{294} = \tfrac{1}{2} \cdot \sqrt{49} \cdot \sqrt{6} = \tfrac{7}{2}\sqrt{6}$$

$$A_{ABCD} = 2\sqrt{6} + \tfrac{7}{2}\sqrt{6} = \underline{\underline{\tfrac{11}{2}\sqrt{6}}}$$

Oppgave 4.84

a) La $\vec{a}$ og $\vec{b}$ være to vektorer som ikke er nullvektor, og la v være vinkelen mellom dem.

$$\vec{a} \parallel \vec{b} \iff v = 0^\circ \text{ eller } v = 180^\circ \iff \sin v = 0 \iff |\vec{a}| \cdot |\vec{b}| \cdot \sin v = 0 \iff |\vec{a} \times \vec{b}| = 0 \iff \vec{a} \times \vec{b} = \vec{0}$$

b)

$$[3,-6,12] \times [-14,28,-56] = \left[\begin{vmatrix} -6 & 12 \\ 28 & -56 \end{vmatrix}, -\begin{vmatrix} 3 & 12 \\ -14 & -56 \end{vmatrix}, \begin{vmatrix} 3 & -6 \\ -14 & 28 \end{vmatrix}\right]$$

$$= \left[336-336, -\left(-168-(-168)\right), 84-84\right] = [0,0,0] = \vec{0}$$

$$\iff \underline{\underline{[3,-6,12] \parallel [-14,28,-56]}}$$

4.9 Volum

Oppgave 4.90

a) Gitt punktene $A(1,1,0)$, $B(4,2,1)$, $D(2,4,2)$ og $E(1,2,2)$

$\Rightarrow$

$$\left.\begin{array}{l}\overrightarrow{AB}=[4-1,2-1,1-0]=[3,1,1]\\ \overrightarrow{AD}=[2-1,4-1,2-0]=[1,3,2]\\ \overrightarrow{AE}=[1-1,2-1,2-0]=[0,1,2]\end{array}\right\} \quad V=\begin{vmatrix}3&1&1\\1&3&2\\0&1&2\end{vmatrix}=3\cdot\begin{vmatrix}3&2\\1&2\end{vmatrix}-1\cdot\begin{vmatrix}1&2\\0&2\end{vmatrix}+1\cdot\begin{vmatrix}1&3\\0&1\end{vmatrix}$$

$$=3\cdot(6-2)-1\cdot(2-0)+1\cdot(1-0)$$

$$=3\cdot4-1\cdot2+1\cdot1=12-2+1=\underline{\underline{11}}$$

b)

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\vec{e}_1&\vec{e}_2&\vec{e}_3\\3&1&1\\1&3&2\end{vmatrix}=\left[\begin{vmatrix}1&1\\3&2\end{vmatrix},-\begin{vmatrix}3&1\\1&2\end{vmatrix},\begin{vmatrix}3&1\\1&3\end{vmatrix}\right]=\left[2-3,-(6-1),9-1\right]=[-1,-5,8]$$

Arealet av grunnflata: $G=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|=\sqrt{(-1)^2+(-5)^2+8^2}=\sqrt{1+25+64}=\sqrt{90}$

$$V=G\cdot h \quad\Leftrightarrow\quad h=\frac{V}{G}=\underline{\underline{\frac{11}{\sqrt{90}}}}$$

Oppgave 4.91

a) Gitt punktene $A(1,1,-1)$, $B(4,2,1)$, $D(0,4,2)$ og $T(1,2,5)$

$\Rightarrow$

$$\overrightarrow{AB}=[4-1,2-1,1-(-1)]=[3,1,2] \qquad \overrightarrow{AD}=[0-1,4-1,2-(-1)]=[-1,3,3]$$

$$\overrightarrow{AT}=[1-1,2-1,5-(-1)]=[0,1,6]$$

$$C(x,y,z) \quad\Rightarrow\quad \overrightarrow{BC}=[x-4,y-2,z-1]$$

$$\overrightarrow{BC}=\overrightarrow{AD} \quad\Rightarrow\quad [x-4,y-2,z-1]=[-1,3,3] \quad\Leftrightarrow\quad x-4=-1 \;\wedge\; y-2=3 \;\wedge\; z-1=3$$

$$\Leftrightarrow\quad x=3 \;\wedge\; y=5 \;\wedge\; z=4 \qquad \underline{\underline{C(3,5,\ 4)}}$$

b)

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\vec{e}_1&\vec{e}_2&\vec{e}_3\\3&1&2\\-1&3&3\end{vmatrix}=\left[\begin{vmatrix}1&2\\3&3\end{vmatrix},-\begin{vmatrix}3&2\\-1&3\end{vmatrix},\begin{vmatrix}3&1\\-1&3\end{vmatrix}\right]=\left[3-6,-(9-(-2)),9-(-1)\right]=[-3,-11,10]$$

Arealet av grunnflata: $G=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|=\sqrt{(-3)^2+(-11)^2+10^2}=\sqrt{9+121+100}=\underline{\underline{\sqrt{230}}}$

c) $V=\frac{1}{3}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AD}\right)\cdot\overrightarrow{AT}=\frac{1}{3}\cdot[-3,-11,10]\cdot[0,1,6]=\frac{1}{3}\cdot(0-11+60)=\underline{\underline{\frac{49}{3}}}$

d) $\overrightarrow{AC}=\left[3-1,5-1,4-(-1)\right]=[2,4,5]$

$$\cos\angle CAT=\frac{\overrightarrow{AC}\cdot\overrightarrow{AT}}{\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{AT}\right|}=\frac{[2,4,5]\cdot[0,1,6]}{\sqrt{2^2+4^2+5^2}\cdot\sqrt{0^2+1^2+6^2}}=\frac{0+4+30}{\sqrt{45}\cdot\sqrt{37}}=\frac{34}{\sqrt{45}\cdot\sqrt{37}}$$

$\Rightarrow\ \underline{\underline{\angle CAT\approx 33,6°}}$

Oppgave 4.92

a) Gitt punktene $A(-1,2,-1)$, $B(3,-1,1)$, $C(1,4,3)$ og $D(1,1,6)$.

$\Rightarrow$

$\overrightarrow{AB}=\left[3-(-1),-1-2,1-(-1)\right]=[4,-3,2]$

$\overrightarrow{AC}=\left[1-(-1),4-2,3-(-1)\right]=[2,2,4]$

$\overrightarrow{AD}=\left[1-(-1),1-2,6-(-1)\right]=[2,-1,7]$

$\overrightarrow{BC}=\left[1-3,4-(-1),3-1\right]=[-2,5,2]$

$\overrightarrow{BD}=\left[1-3,1-(-1),6-1\right]=[-2,2,5]$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\left[\begin{vmatrix}-3&2\\2&4\end{vmatrix},-\begin{vmatrix}4&2\\2&4\end{vmatrix},\begin{vmatrix}4&-3\\2&2\end{vmatrix}\right]=\left[-12-4,-(16-4),8-(-6)\right]=[-16,-12,14]$$

$$V=\frac{1}{6}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD}=\frac{1}{6}\cdot[-16,-12,14]\cdot[2,-1,7]=\frac{1}{6}\cdot(-32+12+98)=\frac{1}{6}\cdot78=\underline{\underline{13}}$$

b) Arealet av grunnflata ABC: $G=\frac{1}{2}\cdot\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\cdot\sqrt{(-16)^2+(-12)^2+14^2}=\frac{1}{2}\cdot\sqrt{596}$

$$V=\frac{1}{3}Gh\quad\Leftrightarrow\quad h=\frac{3V}{G}\quad\Rightarrow\quad h=\frac{3\cdot13}{\frac{1}{2}\cdot\sqrt{596}}=\frac{3\cdot13}{\frac{1}{2}\cdot\sqrt{4}\cdot\sqrt{149}}=\underline{\underline{\frac{39}{\sqrt{149}}}}$$

c) $$\overrightarrow{BC}\times\overrightarrow{BD}=\left[\begin{vmatrix}5&2\\2&5\end{vmatrix},-\begin{vmatrix}-2&2\\-2&5\end{vmatrix},\begin{vmatrix}-2&5\\-2&2\end{vmatrix}\right]=\left[25-4,-(-10+4),-4+10\right]=[21,6,6]$$

Arealet av grunnflata BCD: $G=\frac{1}{2}\cdot\left|\overrightarrow{BC}\times\overrightarrow{BD}\right|=\frac{1}{2}\cdot\sqrt{21^2+6^2+6^2}=\frac{1}{2}\cdot\sqrt{513}=\frac{1}{2}\cdot\sqrt{9}\cdot\sqrt{57}=\frac{3}{2}\sqrt{57}$

$$V=\frac{1}{3}Gh\quad\Leftrightarrow\quad h=\frac{3V}{G}\quad\Rightarrow\quad h=\frac{3\cdot13}{\frac{3}{2}\cdot\sqrt{57}}=\underline{\underline{\frac{26}{\sqrt{57}}}}$$

Oppgave 4.93

a) Vektorene $\vec{a}$, $\vec{b}$ og $\vec{c}$ ligger i samme plan

$\Leftrightarrow$ Volumet av parallellpipedet bestemt av $\vec{a}$, $\vec{b}$ og $\vec{c}$ er 0

$\Leftrightarrow$ Absoluttverdien av determinanten til $\vec{a}$, $\vec{b}$ og $\vec{c}$ er 0

$\Leftrightarrow$ Determinanten til $\vec{a}$, $\vec{b}$ og $\vec{c}$ er 0

b) Gitt punktene $A(1,3,0)$, $B(2,1,-3)$, $C(3,0,4)$ og $D(4,2,1)$

$\Rightarrow$

$$\left.\begin{aligned}\overrightarrow{AB} &= [2-1, 1-3, -3-0] = [1,-2,-3]\\ \overrightarrow{AC} &= [3-1, 0-3, 4-0] = [2,-3,4]\\ \overrightarrow{AD} &= [4-1, 2-3, 1-0] = [3,-1,1]\end{aligned}\right\}$$

$$\begin{vmatrix}1 & -2 & -3\\ 2 & -3 & 4\\ 3 & -1 & 1\end{vmatrix} = 1\cdot\begin{vmatrix}-3 & 4\\ -1 & 1\end{vmatrix} - (-2)\cdot\begin{vmatrix}2 & 4\\ 3 & 1\end{vmatrix} + (-3)\cdot\begin{vmatrix}2 & -3\\ 3 & -1\end{vmatrix} = 1\cdot(-3+4) + 2\cdot(2-12) - 3\cdot(-2+9)$$

$$= 1\cdot 1 + 2\cdot(-10) - 3\cdot 7 = 1 - 20 - 21 = -4($$

Punktene ligger ikke i samme plan.

5.1 Likningen for et plan

Oppgave 5.10

a) $x+3y-3z-4=0 \quad \Rightarrow \quad \underline{\underline{\vec{n}=[1,3,-3]}}$

b) $x-y+z+1=0 \quad \Rightarrow \quad \underline{\underline{\vec{n}=[1,-1,1]}}$

c) $x+2z+6=0 \quad \Rightarrow \quad \underline{\underline{\vec{n}=[1,0,2]}}$

Oppgave 5.11

a) $2x+3y-3z-4=0$

$A(3,1,-3) \quad \Rightarrow \quad 2\cdot 3+3\cdot 1-3\cdot(-3)-4=6+3+9-4=14$

$B(2,-1,-1) \quad \Rightarrow \quad 2\cdot 2+3\cdot(-1)-3\cdot(-1)-4=4-3+3-4=0$

$C(-1,5,2) \quad \Rightarrow \quad 2\cdot(-1)+3\cdot 5-3\cdot 2-4=-2+15-6-4=3$

$\underline{\underline{B \text{ ligger i planet.}}}$

b) $x-y+z+1=0$

$A(3,1,-3) \quad \Rightarrow \quad 3-1+(-3)+1=3-1-3+1=0$

$B(2,-1,-1) \quad \Rightarrow \quad 2-(-1)+(-1)+1=2+1-1+1=3$

$C(-1,5,2) \quad \Rightarrow \quad (-1)-5+2+1=-1-5+2+1=-3$

$\underline{\underline{A \text{ ligger i planet.}}}$

c) $x+2z-3=0$

$A(3,1,-3) \quad \Rightarrow \quad 3+2\cdot(-3)-3=3-6-3=-6$

$B(2,-1,-1) \quad \Rightarrow \quad 2+2\cdot(-1)-3=2-2-3=-3$

$C(-1,5,2) \quad \Rightarrow \quad -1+2\cdot 2-3=-1+4-3=0$

$\underline{\underline{C \text{ ligger i planet.}}}$

Oppgave 5.12

a) $a\cdot(x-x_0)+b\cdot(y-y_0)+c\cdot(z-z_0)=0 \quad \Rightarrow$

$3\cdot(x-2)+1\cdot(y-4)+1\cdot(z-(-3))=0 \quad \Leftrightarrow \quad 3x-6+y-4+z+3=0 \quad \Leftrightarrow \quad \underline{\underline{3x+y+z-7=0}}$

b) $0\cdot(x-(-1))+3\cdot(y-0)+4\cdot(z-4)=0 \quad \Leftrightarrow \quad \underline{\underline{3y+4z-16=0}}$

c) $1\cdot(x-0)+(-1)\cdot(y-0)+1\cdot(z-0)=0 \quad \Leftrightarrow \quad \underline{\underline{x-y+z=0}}$

d) $0\cdot(x-1)+1\cdot(y-2)+0\cdot(z-3)=0 \quad \Leftrightarrow \quad \underline{\underline{y-2=0}}$

Oppgave 5.13

a) Gitt punktene: $A(1,0,1)$, $B(2,5,3)$ og $C(3,4,4)$

$\overrightarrow{AB}=[1,5,2]$, $\overrightarrow{AC}=[2,4,3]$ og $\vec{n}=[7,1,-6]$

$\overrightarrow{AB}\cdot\vec{n}=[1,5,2]\cdot[7,1,-6]=7+5-12=0 \quad \Rightarrow \quad \overrightarrow{AB}\perp\vec{n}$

$\overrightarrow{AC}\cdot\vec{n}=[2,4,3]\cdot[7,1,-6]=14+4-18=0 \quad \Rightarrow \quad \overrightarrow{AC}\perp\vec{n}$

$\left.\begin{array}{l}\overrightarrow{AB} \text{ og } \overrightarrow{AC} \text{ ligger i planet og er ikke parallelle.}\\ \vec{n} \text{ står normalt på begge vektorene.}\end{array}\right\}\quad \underline{\underline{\vec{n}\text{ står vinkelrett på planet.}}}$

b) $7\cdot(x-1)+1\cdot(y-0)+(-6)\cdot(z-1)=0 \quad \Leftrightarrow \quad 7x-7+y-6z+6=0 \quad \Leftrightarrow \quad \underline{\underline{7x+y-6z-1=0}}$

c) $D(4,9,6) \quad \Rightarrow \quad 7\cdot4+9-6\cdot6-1=28+9-36-1=0 \quad$ $\underline{\underline{D \text{ ligger i planet.}}}$

Oppgave 5.14

a) Gitt punktene: $A(1,1,1)$, $B(5,2,3)$ og $C(2,3,3)$

$\overrightarrow{AB}=[4,1,2]$, $\overrightarrow{AC}=[1,2,2]$

$$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\left[\begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix},-\begin{vmatrix}4 & 2\\ 1 & 2\end{vmatrix},\begin{vmatrix}4 & 1\\ 1 & 2\end{vmatrix}\right]=[2-4,-(8-2),8-1]=[-2,-6,7]$$

$-2\cdot(x-1)+(-6)\cdot(y-1)+7\cdot(z-1)=0 \quad \Leftrightarrow \quad -2x+2-6y+6+7z-7=0 \quad \Leftrightarrow$

$\underline{\underline{-2x-6y+7z+1=0}}$

b) Gitt punktene: $A(3,1,2)$, $B(4,1,5)$ og $C(2,2,2)$

$\overrightarrow{AB}=[1,0,3]$, $\overrightarrow{AC}=[-1,1,0]$

$$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\left[\begin{vmatrix}0 & 3\\ 1 & 0\end{vmatrix},-\begin{vmatrix}1 & 3\\ -1 & 0\end{vmatrix},\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}\right]=\left[0-3,-(0+3),1-0\right]=[-3,-3,1]$$

$$-3\cdot(x-3)+(-3)\cdot(y-1)+1\cdot(z-2)=0 \iff -3x+9-3y+3+z-2=0 \iff$$

$$\underline{\underline{-3x-3y+z+10=0}}$$

c) Gitt punktene: $A(4,2,3)$, $B(6,0,7)$ og $C(3,3,1)$

$\overrightarrow{AB}=[2,-2,4]$, $\overrightarrow{AC}=[-1,1,-2]$

$$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\left[\begin{vmatrix}-2 & 4\\ 1 & -2\end{vmatrix},-\begin{vmatrix}2 & 4\\ -1 & -2\end{vmatrix},\begin{vmatrix}2 & -2\\ -1 & 1\end{vmatrix}\right]=\left[4-4,-(-4+4),2-2\right]=[0,0,0]=\vec{0}$$

$\Rightarrow \overrightarrow{AB}\parallel\overrightarrow{AC} \iff$ A , B og C ligger på linje.

<u>Planet er ikke entydig bestemt da normalvektoren er lik nullvektoren.</u>

Oppgave 5.15

Hvis konstantleddet $d=0$, vil planet gå gjennom origo.

5.2 Vinkelen mellom to plan

Oppgave 5.20

a)

$$\alpha: 2x + y + 3z + 2 = 0 \quad \Rightarrow \quad \vec{n_\alpha} = [2,1,3] \quad \Rightarrow \quad \left|\vec{n_\alpha}\right| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{4+1+9} = \sqrt{14}$$

$$\beta: x - 2y + 2z + 1 = 0 \quad \Rightarrow \quad \vec{n_\beta} = [1,-2,2] \quad \Rightarrow \quad = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\cos\angle\left(\vec{n_\alpha}, \vec{n_\beta}\right) = \frac{\vec{n_\alpha}\cdot\vec{n_\beta}}{\left|\vec{n_\alpha}\right|\cdot\left|\vec{n_\beta}\right|} = \frac{[2,1,3]\cdot[1,-2,2]}{\sqrt{14}\cdot 3} = \frac{2-2+6}{\sqrt{14}\cdot 3} = \frac{2}{\sqrt{14}}$$

$$\Rightarrow \quad \angle\left(\vec{n_\alpha}, \vec{n_\beta}\right) \approx 57{,}7^\circ \quad \Rightarrow \quad \underline{\underline{\angle(\alpha,\beta) \approx 57{,}7^\circ}}$$

b)

$$\alpha: x + y + 3z + 4 = 0 \quad \Rightarrow \quad \vec{n_\alpha} = [1,1,3] \quad \Rightarrow \quad \left|\vec{n_\alpha}\right| = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{1+1+9} = \sqrt{11}$$

$$\beta: x - y - 2z + 3 = 0 \quad \Rightarrow \quad \vec{n_\beta} = [1,-1,-2] \quad \Rightarrow \quad = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$\cos\angle\left(\vec{n_\alpha}, \vec{n_\beta}\right) = \frac{\vec{n_\alpha}\cdot\vec{n_\beta}}{\left|\vec{n_\alpha}\right|\cdot\left|\vec{n_\beta}\right|} = \frac{[1,1,3]\cdot[1,-1,-2]}{\sqrt{11}\cdot\sqrt{6}} = \frac{1-1-6}{\sqrt{11}\cdot\sqrt{6}} = \frac{-6}{\sqrt{11}\cdot\sqrt{6}}$$

$$\Rightarrow \quad \angle\left(\vec{n_\alpha}, \vec{n_\beta}\right) \approx 137{,}6^\circ \quad \Rightarrow \quad \angle(\alpha,\beta) \approx 180^\circ - 137{,}6 = \underline{\underline{42{,}4^\circ}}$$

c)

$$\alpha: x + y + 2z + 1 = 0 \quad \Rightarrow \quad \vec{n_\alpha} = [1,1,2] \quad \Rightarrow \quad \left|\vec{n_\alpha}\right| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$\beta: -x - y + z + 3 = 0 \quad \Rightarrow \quad \vec{n_\beta} = [-1,-1,1] \quad \Rightarrow \quad = \sqrt{(-1)^2 + (-1)^2 + 1^2} = \sqrt{3}$$

$$\cos\angle\left(\vec{n_\alpha}, \vec{n_\beta}\right) = \frac{\vec{n_\alpha}\cdot\vec{n_\beta}}{\left|\vec{n_\alpha}\right|\cdot\left|\vec{n_\beta}\right|} = \frac{[1,1,2]\cdot[-1,-1,1]}{\sqrt{6}\cdot\sqrt{3}} = \frac{-1-1+2}{\sqrt{6}\cdot\sqrt{3}} = 0$$

$$\Rightarrow \quad \angle\left(\vec{n_\alpha}, \vec{n_\beta}\right) = 90^\circ \quad \Rightarrow \quad \underline{\underline{\angle(\alpha,\beta) = 90^\circ}}$$

Oppgave 5.21

α: $A(1,1,1)$, $B(4,3,3)$ og $C(2,5,2)$

$\overrightarrow{AB} = [3,2,2]$, $\overrightarrow{AC} = [1,4,1]$

$$\overrightarrow{n_\alpha} = \overrightarrow{AB} \times \overrightarrow{AC} = \left[\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix}, -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} \right] = [2-8, -(3-2), 12-2] = [-6,-1,10]$$

β: $D(-1,1,2)$, $E(2,3,2)$ og $F(1,1,4)$

$\overrightarrow{DE} = [3,2,0]$, $\overrightarrow{DF} = [2,0,2]$

$$\overrightarrow{n_\beta} = \overrightarrow{DE} \times \overrightarrow{DF} = \left[\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}, -\begin{vmatrix} 3 & 0 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} \right] = [4-0, -(6-0), 0-4] = [4,-6,-4]$$

$$\left|\overrightarrow{n_\alpha}\right| = \sqrt{(-6)^2 + (-1)^2 + 10^2} = \sqrt{137} \qquad \left|\overrightarrow{n_\beta}\right| = \sqrt{4^2 + (-6)^2 + (-4)^2} = \sqrt{68}$$

$$\cos\angle\left(\overrightarrow{n_\alpha}, \overrightarrow{n_\beta}\right) = \frac{\overrightarrow{n_\alpha} \cdot \overrightarrow{n_\beta}}{\left|\overrightarrow{n_\alpha}\right| \cdot \left|\overrightarrow{n_\beta}\right|} = \frac{[-6,-1,10]\cdot[4,-6,-4]}{\sqrt{137}\cdot\sqrt{68}} = \frac{-24+6-40}{\sqrt{137}\cdot\sqrt{68}} = \frac{-58}{\sqrt{137}\cdot\sqrt{68}}$$

$$\Rightarrow \quad \angle\left(\overrightarrow{n_\alpha}, \overrightarrow{n_\beta}\right) \approx 126{,}9° \quad \Rightarrow \quad \underline{\underline{\angle(\alpha,\beta) = 53{,}1°}}$$

Oppgave 5.22

La $\overrightarrow{n_\alpha}$ og $\overrightarrow{n_\beta}$ være normalvektorene til to plan α og β, og la v være vinkelen mellom vektorene.

$$\alpha \parallel \beta \iff \overrightarrow{n_\alpha} \parallel \overrightarrow{n_\beta} \iff v = 0° \text{ eller } v = 180° \iff \sin v = 0 \iff \left|\overrightarrow{n_\alpha}\right| \cdot \left|\overrightarrow{n_\beta}\right| \cdot \sin v = 0$$

$$\iff \left|\overrightarrow{n_\alpha} \times \overrightarrow{n_\beta}\right| = 0 \iff \overrightarrow{n_\alpha} \times \overrightarrow{n_\beta} = \vec{0}$$

5.3 Rette linjer i rommet

Oppgave 5.30

a)

Gitt punktet $P(2,3,-1)$ og retningsvektoren $\vec{r} = [2,1,-3] \quad \Rightarrow \quad l: \begin{cases} x = 2+2t \\ y = 3+t \\ z = -1-3t \end{cases}$

b)

$x-2y+3z-2=0 \quad \Leftrightarrow \quad (2+2t)-2\cdot(3+t)+3\cdot(-1-3t)-2=0 \quad \Leftrightarrow$

$2+2t-6-2t-3-9t-2=0 \quad \Leftrightarrow \quad -9t=9 \quad \Leftrightarrow \quad t=-1$

$$\Rightarrow \begin{array}{l} x = 2+2\cdot(-1)=0 \\ y=3+(-1)=2 \\ z=-1-3\cdot(-1)=2 \end{array} \qquad \text{Skjæringspunkt } (0,2,2)$$

Oppgave 5.31

a) Gitt punktet $P(2,0,3)$ og planet $\alpha: x+2y-2z+13=0 \quad \Rightarrow \vec{n}_\alpha = [1,2,-2]$

$l: \begin{cases} x = 2+t \\ y = 2t \\ z = 3-2t \end{cases}$

b)

$x+2y-2z+13=0 \quad \Leftrightarrow \quad (2+t)+2\cdot 2t-2\cdot(3-2t)+13=0 \quad \Leftrightarrow$

$2+t+4t-6+4t+13=0 \quad \Leftrightarrow \quad 9t=-9 \quad \Leftrightarrow \quad t=-1$

$$\Rightarrow \begin{array}{l} x = 2+(-1)=1 \\ y=2\cdot(-1)=-2 \\ z=3-2\cdot(-1)=5 \end{array} \qquad \text{Skjæringspunkt Q}(1,-2,5)$$

c) Avstanden mellom P og α tilsvarer lengden av $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = [-1,-2,2] \quad \Rightarrow \quad d = \sqrt{(-1)^2+(-2)^2+2^2} = \sqrt{9} = 3$

Oppgave 5.32

Gitt punktet $P(1,0,-2)$ og retningsvektoren $\vec{r}=[4,3,3] \quad \Rightarrow \quad l:\begin{cases} x=1+4t \\ y=3t \\ z=-2+3t \end{cases}$

Søkte punkter har koordinater (x,y,z) med avstand 3 til punktet $(7,4,3) \quad \Rightarrow$

$$\left|[x-7,y-4,z-3]\right|=3 \iff \sqrt{(x-7)^2+(y-4)^2+(z-3)^2}=3 \iff (x-7)^2+(y-4)^2+(z-3)^2=9 \iff$$

$$(1+4t-7)^2+(3t-4)^2+(-2+3t-3)^2=9 \iff (4t-6)^2+(3t-4)^2+(3t-5)^2=9 \iff$$

$$16t^2-48t+36+9t^2-24t+16+9t^2-30t+25-9=0 \iff 34t^2-102t+68=0 \overset{:34}{\iff} t^2-3t+2=0 \iff$$

$$t=\frac{-(-3)\pm\sqrt{(-3)^2-4\cdot1\cdot2}}{2\cdot1}=\frac{3\pm\sqrt{1}}{2} \iff t=2 \quad \vee \quad t=1$$

$$t=2 \quad \Rightarrow \quad \begin{cases} x=1+4\cdot2=9 \\ y=3\cdot2=6 \\ z=-2+3\cdot2=4 \end{cases} \qquad t=1 \quad \Rightarrow \quad \begin{cases} x=1+4\cdot1=5 \\ y=3\cdot1=3 \\ z=-2+3\cdot1=1 \end{cases}$$

De to punktene er $(9,6,4)$ og $(5,3,1)$.

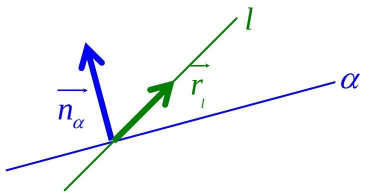
Oppgave 5.33

Gitt linjene $l:\begin{cases} x=-1+t \\ y=-2+t \\ z=2+2t \end{cases}$ og $m:\begin{cases} x=-2+t \\ y=1+2t \\ z=3-2t \end{cases} \quad \Rightarrow \quad \vec{r_l}=[1,1,2]$ og $\vec{r_m}=[1,2,-2]$

$$\cos\angle\left(\vec{r_l},\vec{r_m}\right)=\frac{\vec{r_l}\cdot\vec{r_m}}{\left|\vec{r_l}\right|\cdot\left|\vec{r_m}\right|}=\frac{[1,1,2]\cdot[1,2,-2]}{\sqrt{1^2+1^2+2^2}\cdot\sqrt{1^2+2^2+(-2)^2}}=\frac{1+2-4}{\sqrt{6}\cdot3}=-\frac{1}{\sqrt{6}\cdot3}$$

$$\Rightarrow \quad \angle\left(\vec{r_l},\vec{r_m}\right)\approx 97{,}8° \quad \Rightarrow \quad \angle(l,m)\approx 180°-97{,}8°=\underline{\underline{82{,}2°}}$$

Oppgave 5.34



Gitt linja $l:\begin{cases} x=-1+t \\ y=-2+t \\ z=2+2t \end{cases}$ og planet $\alpha: 2x-y+2z-1=0$

$\Rightarrow \quad \vec{r_l}=[1,1,2]$ og $\vec{n_\alpha}=[2,-1,2]$

$$\cos\angle\left(\vec{r_l},\vec{n_\alpha}\right)=\frac{\vec{r_l}\cdot\vec{n_\alpha}}{\left|\vec{r_l}\right|\cdot\left|\vec{n_\alpha}\right|}=\frac{[1,1,2]\cdot[2,-1,2]}{\sqrt{1^2+1^2+2^2}\cdot\sqrt{2^2+(-1)^2+2^2}}$$

$$=\frac{2-1+4}{\sqrt{6}\cdot 3}=\frac{5}{\sqrt{6}\cdot 3}$$

$$\Rightarrow \angle\left(\vec{r_l},\vec{n_\alpha}\right)\approx 47{,}1^\circ \qquad \Rightarrow \qquad \angle(l,\alpha)\approx 90^\circ-47{,}1^\circ=\underline{\underline{42{,}9^\circ}}$$

Oppgave 5.35

a) Gitt punktene $A(1,1,0)$, $B(4,0,1)$, $C(2,3,2)$ og $D(3,1,5)$

$\overrightarrow{AB}=[3,-1,1] \qquad \overrightarrow{AC}=[1,2,2]$

$$\vec{n_\alpha}=\overrightarrow{AB}\times\overrightarrow{AC}=\left[\begin{vmatrix}-1 & 1\\ 2 & 2\end{vmatrix},-\begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix},\begin{vmatrix}3 & -1\\ 1 & 2\end{vmatrix}\right]=\left[-2-2,-(6-1),6+1\right]=[-4,-5,7]$$

$$-4\cdot(x-1)-5\cdot(y-1)+7\cdot(z-0)=0 \iff -4x+4-5y+5+7z=0 \iff \underline{\underline{\alpha:-4x-5y+7z+9=0}}$$

b)

$$\vec{r_l}=\overrightarrow{AD}=[2,0,5] \quad \Rightarrow \quad \underline{\underline{l:\begin{cases} x=1+2t \\ y=1 \\ z=5t \end{cases}}}$$

c)

$$\cos\angle\left(\vec{r_l},\vec{n_\alpha}\right)=\frac{\vec{r_l}\cdot\vec{n_\alpha}}{\left|\vec{r_l}\right|\cdot\left|\vec{n_\alpha}\right|}=\frac{[2,0,5]\cdot[-4,-5,7]}{\sqrt{2^2+0^2+5^2}\cdot\sqrt{(-4)^2+(-5)^2+7^2}}=\frac{-8-0+35}{\sqrt{29}\cdot\sqrt{90}}=\frac{27}{\sqrt{29}\cdot\sqrt{90}}$$

$$\Rightarrow \angle\left(\vec{r_l},\vec{n_\alpha}\right)\approx 58{,}1^\circ \qquad \Rightarrow \qquad \angle(l,\alpha)\approx 90^\circ-58{,}1^\circ=\underline{\underline{31{,}9^\circ}}$$

5.4 Parameterframstilling for et plan

Oppgave 5.40

a) Planet α går gjennom punktene $A(2,1,0)$, $B(4,3,1)$ og $C(3,4,2)$

$$\overrightarrow{AB}=[2,2,1] \quad \overrightarrow{AC}=[1,3,2] \quad \Rightarrow \quad \overrightarrow{AB} \nparallel \overrightarrow{AC} \quad \Rightarrow \quad \alpha:\begin{cases} x=2+2t+s \\ y=1+2t+3s \\ z=0+t+2s \end{cases}$$

b)

Punktet $(5,10,6)$ ligger i planet $\Rightarrow \begin{cases} 5=2+2t+s \\ 10=1+2t+3s \\ 6=0+t+2s \end{cases} \Leftrightarrow \begin{cases} s=3-2t \ \ (1) \\ 2t+3s=9 \ \ (2) \\ t+2s=6 \ \ (3) \end{cases} \overset{1 \text{ inn i } 3}{\Leftrightarrow}$

$$t+2\cdot(3-2t)=6 \Leftrightarrow t+6-4t=6 \Leftrightarrow -3t=0 \Leftrightarrow t=0 \Rightarrow s=3-2\cdot 0=3$$

Sjekker om $t=0$ og $s=3$ også passer i (2): $2\cdot 0+3\cdot 3=9$ Punktet $(5,10,6)$ ligger i planet.

c)

Punktet $(6,2,0)$ ligger i planet $\Rightarrow \begin{cases} 6=2+2t+s \\ 2=1+2t+3s \\ 0=0+t+2s \end{cases} \Leftrightarrow \begin{cases} 2t+s=4 \ \ (1) \\ 2t+3s=1 \ \ (2) \\ t=-2s \ \ (3) \end{cases} \overset{3 \text{ inn i } 1}{\Leftrightarrow}$

$$2\cdot(-2s)+s=4 \Leftrightarrow -3s=4 \Leftrightarrow s=-\tfrac{4}{3} \Rightarrow t=-2\cdot\left(-\tfrac{4}{3}\right)=\tfrac{8}{3}$$

Sjekker om $t=\frac{8}{3}$ og $s=-\frac{4}{3}$ også passer i (2): $2\cdot\frac{8}{3}+3\cdot\left(-\frac{4}{3}\right)=\frac{16}{3}-\frac{12}{3}=\frac{4}{3}\neq 1$

Punktet $(6,2,0)$ ligger ikke i planet.

Oppgave 5.41

a)

Gitt punktet $A(2,4,1)$ og linja $l:\begin{cases} x=1-t \\ y=6+5t \\ z=2+3t \end{cases}$

$t=0 \Rightarrow B(1,\ 6,\ 2)$ er et punkt som ligger på linja α, og derfor i planet.

$\vec{r}=[-1,\ 5,\ 3]$ og $\overrightarrow{AB}=[-1,\ 2,\ 1]$ er ikke parallelle og ligger begge i planet $\Rightarrow \alpha:\begin{cases} x=2-t-s \\ y=4+5t+2s \\ z=1+3t+s \end{cases}$

b)

Punktet $(5,\ 10,\ 6)$ ligger i planet $\Rightarrow \begin{cases} 5 = 2 - t - s \\ 10 = 4 + 5t + 2s \\ 6 = 1 + 3t + s \end{cases} \Leftrightarrow \begin{cases} t + s = -3 \ \ (1) \\ 5t + 2s = 6 \ \ (2) \\ s = -3t + 5 \ \ (3) \end{cases} \overset{\text{3 inn i 2}}{\Leftrightarrow}$

$5t + 2(-3t + 5) = 6 \Leftrightarrow -t = -4 \Leftrightarrow t = 4 \Rightarrow s = -3 \cdot 4 + 5 = -7$

Sjekker om $t = 4$ og $s = -7$ også passer i (1): $4 - 7 = -3$ $\underline{\underline{\text{Punktet } (5,\ 10,\ 6) \text{ ligger i planet.}}}$

Oppgave 5.42

a) Gitt planet $\alpha : x + 3y + 2z - 6 = 0$

Innfører parameterne t og s slik at $y = t \ \wedge \ z = s \Rightarrow x + 3t + 2s - 6 = 0 \Leftrightarrow x = 6 - 3t - 2s$

$$\underline{\underline{\alpha : \begin{cases} x = 6 - 3t - 2s \\ y = t \\ z = s \end{cases}}}$$

b) Gitt planet $\alpha : 2x + y + 3z + 4 = 0$

Innfører parameterne t og s slik at $x = t \ \wedge \ z = s \Rightarrow 2t + y + 3s + 4 = 0 \Leftrightarrow y = -4 - 2t - 3s$

$$\underline{\underline{\alpha : \begin{cases} x = t \\ y = -4 - 2t - 3s \\ z = s \end{cases}}}$$

c) Gitt planet $\alpha : 3x + 5y + 2z - 4 = 0$

Innfører parameterne t og s slik at $x = 2t \ \wedge \ y = 2s \Rightarrow 3 \cdot 2t + 5 \cdot 2s + 2z - 4 = 0 \Leftrightarrow$

$2z = 4 - 6t - 10s \Leftrightarrow z = 2 - 3t - 5s$

$$\underline{\underline{\alpha : \begin{cases} x = 2t \\ y = 2s \\ z = 2 - 3t - 5s \end{cases}}}$$

Oppgave 5.43

a)

Planet $\alpha:\begin{cases} x = 3+2s+3t \\ y = 2+s+t \\ z = 1-2s+t \end{cases}$ går gjennom punktet $(3,2,1)$ og er parallelt med vektorene $\vec{r_1} = [2,1,-2]$ og $\vec{r_2} = [3,1,1]$

$$\vec{n} = \vec{r_1} \times \vec{r_2} = \left[\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}\right] = [1+2, -(2+6), 2-3] = [3,-8,-1]$$

$$3\cdot(x-3) - 8\cdot(y-2) - 1\cdot(z-1) = 0 \iff 3x - 9 - 8y + 16 - z + 1 = 0 \iff$$

$$\underline{\underline{\alpha : 3x - 8y - z + 8 = 0}}$$

b)

$$l \perp \alpha \quad \Rightarrow \quad \vec{r_l} = \vec{n_\alpha} = [3,-8,-1] \quad \Rightarrow \quad l:\begin{cases} x = 1+3t \\ y = 1-8t \\ z = 0-t \end{cases}$$

c) Skjæringspunktet mellom planet og linja er bestemt av at

$$3\cdot(1+3t) - 8\cdot(1-8t) - (-t) + 8 = 0 \iff 3 + 9t - 8 + 64t + t + 8 = 0 \iff 74t = -3 \iff t = -\tfrac{3}{74}$$

$$x = 1 + 3\cdot\left(-\tfrac{3}{74}\right) = \tfrac{65}{74}$$

$$y = 1 - 8\cdot\left(-\tfrac{3}{74}\right) = \tfrac{98}{74} = \tfrac{49}{37}$$

$$z = 0 - \left(-\tfrac{3}{74}\right) = \tfrac{3}{74}$$

Skjæringspunkt i $\underline{\underline{\left(\tfrac{65}{74}, \tfrac{49}{37}, \tfrac{3}{74}\right)}}$.

5.5 Likningen for ei kule

Oppgave 5.50

Kula har radius 3 og sentrum i $(-2,2,1)$ $\Rightarrow$

$(x-(-2))^2+(y-2)^2+(z-1)^2=3^2 \quad \Leftrightarrow \quad \underline{\underline{(x+2)^2+(y-2)^2+(z-1)^2=9}}$

Oppgave 5.51

a) Gitt kula $x^2+y^2+z^2+4x-6y-12=0 \quad \Leftrightarrow \quad x^2+4x+y^2-6y+z^2-12=0$

Lager fullstendige kvadrater:

$x^2+4x+\left(\frac{4}{2}\right)^2+y^2-6y+\left(\frac{6}{2}\right)^2+z^2=12+\left(\frac{4}{2}\right)^2+\left(\frac{6}{2}\right)^2 \quad \Leftrightarrow$

$(x+2)^2+(y-3)^2+(z-0)^2=12+4+9 \quad \Leftrightarrow$

$(x-(-2))^2+(y-3)^2+(z-0)^2=5^2$ $\quad$ $\underline{\underline{\text{Kula har radius 5 og sentrum i } (-2,3,0).}}$

b) Gitt kula $x^2+y^2+z^2-4x+4y-2z=0 \quad \Leftrightarrow \quad x^2-4x+y^2+4y+z^2-2z=0$

Lager fullstendige kvadrater:

$x^2-4x+\left(\frac{4}{2}\right)^2+y^2+4y+\left(\frac{4}{2}\right)^2+z^2-2z+\left(\frac{2}{2}\right)^2=0+\left(\frac{4}{2}\right)^2+\left(\frac{4}{2}\right)^2+\left(\frac{2}{2}\right)^2 \quad \Leftrightarrow$

$(x-2)^2+(y+2)^2+(z-1)^2=4+4+1 \quad \Leftrightarrow$

$(x-2)^2+(y-(-2))^2+(z-1)^2=3^2$ $\quad$ $\underline{\underline{\text{Kula har radius 3 og sentrum i } (2,-2,1).}}$

Oppgave 5.52

a) Linja l går gjennom punktet (4,2,0) og har retningsvektor en $[4,3,3]$ $\Rightarrow$ $\underline{\underline{l:\begin{cases} x=4+4t \\ y=2+3t \\ z=3t \end{cases}}}$

b) Kula K har sentrum (6,3,2) og radius 3 $\Rightarrow$ $\underline{\underline{(x-6)^2+(y-3)^2+(z-2)^2=3^2}}$

c) Skjæringspunkter mellom K og l bestemt ved

$$(4+4t-6)^2+(2+3t-3)^2+(3t-2)^2=9 \iff (4t-2)^2+(3t-1)^2+(3t-2)^2=9 \iff$$
$$16t^2-16t+4+9t^2-6t+1+9t^2-12t+4-9=0 \iff 34t^2-34t=0 \iff 34t\cdot(t-1)=0 \iff$$
$$t=0 \quad \vee \quad t=1$$

$$\begin{cases} x=4+4\cdot 0=4 \\ y=2+3\cdot 0=2 \\ z=3\cdot 0=0 \end{cases} \quad \vee \quad \begin{cases} x=4+4\cdot 1=8 \\ y=2+3\cdot 1=5 \\ z=3\cdot 1=3 \end{cases}$$

Skjæringspunkter $(4,2,0)$ og $(8,5,3)$.

Oppgave 5.53

a) Planet $\alpha: 2x+y+2z-6=0$ har en normalvektor $\overrightarrow{n_\alpha}=[2,1,2]$

Linja l står vinkelrett på $\alpha \Rightarrow \overrightarrow{r_l}=\overrightarrow{n_\alpha}=[2,1,2] \Rightarrow l:\begin{cases} x=5+2t \\ y=4+t \\ z=5+2t \end{cases}$

b) Skjæringspunkter mellom α og l bestemt ved

$$2\cdot(5+2t)+(4+t)+2\cdot(5+2t)-6=0 \iff 10+4t+4+t+10+4t-6=0 \iff$$
$$9t=-18 \iff t=-2$$

$$\begin{cases} x=5+2\cdot(-2)=1 \\ y=4+(-2)=2 \\ z=5+2\cdot(-2)=1 \end{cases}$$

Skjæringspunkt i $(1,2,1)$.

c) Avstanden gitt ved lengden av vektoren mellom (5,4,5) og (1,2,1).

$$d=\left|[4,2,4]\right|=\sqrt{4^2+2^2+4^2}=\sqrt{16+4+16}=\sqrt{36}=\underline{\underline{6}}$$

d) Sentrum i kula er (5,4,5) og radius tilsvarer avstanden fra dette punktet til planet.

$$\underline{\underline{(x-5)^2+(y-4)^2+(z-5)^2=6^2}}$$

Oppgave 5.54

a) Linja går gjennom punktene $A(1,0,4)$ og $B(3,8,2)$.

$\overrightarrow{AB} = [3-1, 8-0, 2-4] = [2, 8, -2] \quad \Rightarrow \quad \vec{r}_l = [1, 4, -1]$

$$l: \begin{cases} x = 1+t \\ y = 4t \\ z = 4-t \end{cases}$$

b) Søkte punkter $Q(x,y,z) \quad \Rightarrow$

$\overrightarrow{PQ} = [x-(-2), y-0, z-1] = [1+t+2, 4t, 4-t-1] = [t+3, 4t, 3-t]$

$|\overrightarrow{PQ}| = 6 \quad \Rightarrow$

$\sqrt{(t+3)^2 + (4t)^2 + (3-t)^2} = 6 \quad \Leftrightarrow \quad (t^2+6t+9) + 16t^2 + (9-6t+t^2) = 6^2 \quad \Leftrightarrow$

$t^2 + 6t + 9 + 16t^2 + 9 - 6t + t^2 - 36 = 0 \quad \Leftrightarrow \quad 18t^2 - 18 = 0 \quad \Leftrightarrow \quad t^2 = 1 \quad \Leftrightarrow \quad t = -1 \ \vee \ t = 1$

$$\begin{cases} x = 1+1 = 2 \\ y = 4\cdot 1 = 4 \\ z = 4-1 = 3 \end{cases} \quad \vee \quad \begin{cases} x = 1-1 = 0 \\ y = 4\cdot(-1) = -4 \\ z = 4-(-1) = 5 \end{cases}$$

Punktene har koordinatene $(2,4,3)$ og $(0,-4,5)$.

5.6 Sfærisk avstand

Oppgave 5.60

a) Sværisk avstand gitt ved $b = v \cdot r \quad \Rightarrow \quad b = \frac{60°}{180°} \cdot \pi \cdot 12 = \underline{\underline{4\pi}}$

b) $b = \frac{120°}{180°} \cdot \pi \cdot 12 = \underline{\underline{8\pi}}$

c) $b = \frac{45°}{180°} \cdot \pi \cdot 12 = \underline{\underline{3\pi}}$

Oppgave 5.61

a) $r = \left|\overrightarrow{SP}\right| = \left|[0,3,0]\right| = \sqrt{0^2 + 3^2 + 0^2} = \underline{\underline{3}}$

$$\cos\angle\left(\overrightarrow{SP}, \overrightarrow{SQ}\right) = \frac{\overrightarrow{SP} \cdot \overrightarrow{SQ}}{\left|\overrightarrow{SP}\right| \cdot \left|\overrightarrow{SQ}\right|} = \frac{[0,3,0] \cdot [1,2,2]}{3 \cdot 3} = \frac{6}{9} \quad \Rightarrow \quad \angle\left(\overrightarrow{SP}, \overrightarrow{SQ}\right) \approx 48,2°$$

Sfærisk avstand mellom P og Q: $\frac{48,2°}{180°} \cdot \pi \cdot 3 \approx \underline{\underline{2,52}}$

b) $r = \left|\overrightarrow{SP}\right| = \left|[3,4,12]\right| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = \underline{\underline{13}}$

$$\cos\angle\left(\overrightarrow{SP}, \overrightarrow{SQ}\right) = \frac{\overrightarrow{SP} \cdot \overrightarrow{SQ}}{\left|\overrightarrow{SP}\right| \cdot \left|\overrightarrow{SQ}\right|} = \frac{[3,4,12] \cdot [5,12,0]}{13 \cdot 13} = \frac{15+48}{169} = \frac{63}{169} \quad \Rightarrow \quad \angle\left(\overrightarrow{SP}, \overrightarrow{SQ}\right) \approx 68,1°$$

Sfærisk avstand mellom P og Q: $\frac{68,1°}{180°} \cdot \pi \cdot 13 \approx \underline{\underline{15,5}}$

c) $r = \left|\overrightarrow{SP}\right| = \left|[5,10,10]\right| = \sqrt{5^2 + 10^2 + 10^2} = \sqrt{225} = \underline{\underline{15}}$

$$\cos\angle\left(\overrightarrow{SP}, \overrightarrow{SQ}\right) = \frac{\overrightarrow{SP} \cdot \overrightarrow{SQ}}{\left|\overrightarrow{SP}\right| \cdot \left|\overrightarrow{SQ}\right|} = \frac{[5,10,10] \cdot [9,12,0]}{15 \cdot 15} = \frac{45+120}{225} = \frac{165}{225} \quad \Rightarrow \quad \angle\left(\overrightarrow{SP}, \overrightarrow{SQ}\right) \approx 42,8°$$

Sfærisk avstand mellom P og Q: $\frac{42,8°}{180°} \cdot \pi \cdot 15 \approx \underline{\underline{11,2}}$

5.7 Sfæriske tokanter og trekanter

Oppgave 5.70

a) Kula med sentrum i $S(0,0,0)$ og punktene $A(0,0,5)$, $B(4,3,0)$ og $C(0,-4,3)$ på denne kula.

$\Rightarrow \ \overrightarrow{SA}=[0,0,5] \ , \ \overrightarrow{SB}=[4,3,0] \ , \ \overrightarrow{SC}=[0,-4,3] \ , \ |\overrightarrow{SA}|=|\overrightarrow{SB}|=|\overrightarrow{SC}|=5$

Finner en normalvektor til planet gjennom A, S og B:

$$\vec{n_1}=\overrightarrow{SA}\times\overrightarrow{SB}=\left[\begin{vmatrix}0 & 5\\3 & 0\end{vmatrix},-\begin{vmatrix}0 & 5\\4 & 0\end{vmatrix},\begin{vmatrix}0 & 0\\4 & 3\end{vmatrix}\right]=\left[0-15,-(0-20),0-0\right]=[-15,20,0]$$

$$|\vec{n_1}|=\sqrt{(-15)^2+20^2+0^2}=\sqrt{625}=25$$

Finner på samme måte en normalvektor til planet gjennom A, S og C:

$$\vec{n_2}=\overrightarrow{SA}\times\overrightarrow{SC}=\left[\begin{vmatrix}0 & 5\\-4 & 3\end{vmatrix},-\begin{vmatrix}0 & 5\\0 & 3\end{vmatrix},\begin{vmatrix}0 & 0\\0 & -4\end{vmatrix}\right]=\left[0+20,-(0-0),0-0\right]=[20,0,0]$$

$$|\vec{n_2}|=\sqrt{20^2+0^2+0^2}=20$$

$$\cos\angle A=\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}=\frac{[-15,20,0]\cdot[20,0,0]}{25\cdot 20}=\frac{-300}{500}=-\frac{3}{5} \quad\Rightarrow\quad \underline{\underline{\angle A\approx 126{,}9^\circ}}$$

b) Arealet av tokanten: $\frac{126{,}9}{90}\cdot\pi\cdot 5^2\approx \underline{\underline{110{,}7}}$

Oppgave 5.71

a) Når punktet $A(4,3,12)$ speiles om origo, får speilbildet koordinatene $\underline{\underline{A'(-4,-3,-12)}}$

b) Kula med sentrum i $S(0,0,0)$ og punktene $A(4,3,12)$, $B(12,5,0)$ og $C(4,12,3)$ på denne kula.

$\Rightarrow \ \overrightarrow{SA}=[4,3,12] \ , \ \overrightarrow{SB}=[12,5,0] \ , \ \overrightarrow{SC}=[4,12,3] \ , \ |\overrightarrow{SA}|=|\overrightarrow{SB}|=|\overrightarrow{SC}|=13$

Finner en normalvektor til planet gjennom A, S og B:

$$\vec{n_1}=\overrightarrow{SA}\times\overrightarrow{SB}=\left[\begin{vmatrix}3 & 12\\5 & 0\end{vmatrix},-\begin{vmatrix}4 & 12\\12 & 0\end{vmatrix},\begin{vmatrix}4 & 3\\12 & 5\end{vmatrix}\right]=\left[0-60,-(0-144),20-36\right]=[-60,144,-16]$$

$$|\vec{n_1}|=\sqrt{(-60)^2+144^2+(-16)^2}=\sqrt{24592}$$

Finner på samme måte en normalvektor til planet gjennom A, S og C:

$$\vec{n_2}=\overrightarrow{SA}\times\overrightarrow{SC}=\left[\begin{vmatrix}3 & 12\\12 & 3\end{vmatrix},-\begin{vmatrix}4 & 12\\4 & 3\end{vmatrix},\begin{vmatrix}4 & 3\\4 & 12\end{vmatrix}\right]=\left[9-144,-(12-48),48-12\right]=[-135,36,36]$$

$$|\vec{n_2}|=\sqrt{(-135)^2+36^2+36^2}=\sqrt{20817}$$

$$\cos\angle A=\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}=\frac{[-60,144,-16]\cdot[-135,36,36]}{\sqrt{24592}\cdot\sqrt{20817}}=\frac{8100+5184-576}{\sqrt{24592}\cdot\sqrt{20817}}=\frac{12708}{\sqrt{24592}\cdot\sqrt{20817}}$$

$\Rightarrow \ \underline{\underline{\angle A\approx 55{,}8^\circ}}$

c) Arealet av tokanten: $\frac{55{,}8}{90}\cdot\pi\cdot 13^2\approx\underline{\underline{329}}$

Oppgave 5.72

a) Kula med sentrum i $S(0,0,0)$ og punktene $A(0,0,5)$, $B(4,3,0)$ og $C(0,-4,3)$ på denne kula.

$$\Rightarrow \ \overrightarrow{SA}=[0,0,5]\ ,\ \ \overrightarrow{SB}=[4,3,0]\ ,\ \ \overrightarrow{SC}=[0,-4,3]\ ,\ \ |\overrightarrow{SA}|=|\overrightarrow{SB}|=|\overrightarrow{SC}|=5$$

$$\vec{n_1}=\overrightarrow{SA}\times\overrightarrow{SB}=\left[\begin{vmatrix}0&5\\3&0\end{vmatrix},-\begin{vmatrix}0&5\\4&0\end{vmatrix},\begin{vmatrix}0&0\\4&3\end{vmatrix}\right]=\left[0-15,-(0-20),0-0\right]=[-15,20,0]$$

$$|\vec{n_1}|=\sqrt{(-15)^2+20^2+0^2}=\sqrt{625}=25$$

$$\vec{n_2}=\overrightarrow{SA}\times\overrightarrow{SC}=\left[\begin{vmatrix}0&5\\-4&3\end{vmatrix},-\begin{vmatrix}0&5\\0&3\end{vmatrix},\begin{vmatrix}0&0\\0&-4\end{vmatrix}\right]=\left[0+20,-(0-0),0-0\right]=[20,0,0]\ \Rightarrow\ |\vec{n_2}|=20$$

$$\cos\angle A=\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}=\frac{[-15,20,0]\cdot[20,0,0]}{25\cdot 20}=\frac{-300}{500}=-\frac{3}{5}\quad\Rightarrow\quad \underline{\underline{\angle A\approx 126{,}9^\circ}}$$

$$\vec{n_3}=\overrightarrow{SB}\times\overrightarrow{SA}=-\overrightarrow{SA}\times\overrightarrow{SC}=-[-15,20,0]=[15,-20,0]$$

$$|\vec{n_3}|=\sqrt{15^2+(-20)^2+0^2}=\sqrt{625}=25$$

$$\vec{n_4}=\overrightarrow{SB}\times\overrightarrow{SC}=\left[\begin{vmatrix}3&0\\-4&3\end{vmatrix},-\begin{vmatrix}4&0\\0&3\end{vmatrix},\begin{vmatrix}4&3\\0&-4\end{vmatrix}\right]=\left[9-0,-(12-0),-16-0\right]=[9,-12,-16]$$

$$|\vec{n_4}|=\sqrt{9^2+(-12)^2+(-16)^2}=\sqrt{481}$$

$$\cos\angle B=\frac{\vec{n_3}\cdot\vec{n_4}}{|\vec{n_3}|\cdot|\vec{n_4}|}=\frac{[15,-20,0]\cdot[9,-12,-16]}{25\cdot\sqrt{481}}=\frac{135+240}{25\cdot\sqrt{481}}=\frac{15}{\sqrt{481}}\quad\Rightarrow\quad \underline{\underline{\angle B\approx 46{,}8^\circ}}$$

$$\vec{n_5}=\overrightarrow{SC}\times\overrightarrow{SA}=-\overrightarrow{SA}\times\overrightarrow{SC}=-[20,0,0]=[-20,0,0]\quad\Rightarrow\quad |\vec{n_5}|=20$$

$$\vec{n_6}=\overrightarrow{SC}\times\overrightarrow{SB}=-\overrightarrow{SB}\times\overrightarrow{SC}=-[9,-12,-16]=[-9,12,16]$$

$$|\vec{n_6}|=\sqrt{481}$$

$$\cos\angle C=\frac{\vec{n_5}\cdot\vec{n_6}}{|\vec{n_5}|\cdot|\vec{n_6}|}=\frac{[-20,0,0]\cdot[-9,12,16]}{20\cdot\sqrt{481}}=\frac{180}{20\cdot\sqrt{481}}=\frac{9}{\sqrt{481}}\quad\Rightarrow\quad \underline{\underline{\angle C\approx 65{,}8^\circ}}$$

b)
$$A=\left(\frac{A+B+C}{180}-1\right)\cdot\pi r^2=\left(\frac{126{,}9+46{,}8+65{,}8}{180}-1\right)\cdot\pi\cdot 5^2\approx\underline{\underline{26{,}0}}$$

Oppgave 5.73

a) Kula med sentrum i $S(0,0,0)$ og punktene $A(4,3,12)$, $B(12,5,0)$ og $C(4,12,3)$ på denne kula.

$\Rightarrow\ \overrightarrow{SA}=[4,3,12]\ ,\ \overrightarrow{SB}=[12,5,0]\ ,\ \overrightarrow{SC}=[4,12,3]\ ,\ |\overrightarrow{SA}|=|\overrightarrow{SB}|=|\overrightarrow{SC}|=13$

$$\vec{n_1}=\overrightarrow{SA}\times\overrightarrow{SB}=\left[\begin{vmatrix}3 & 12\\5 & 0\end{vmatrix},-\begin{vmatrix}4 & 12\\12 & 0\end{vmatrix},\begin{vmatrix}4 & 3\\12 & 5\end{vmatrix}\right]=[0-60,-(0-144),20-36]=[-60,144,-16]$$

$$|\vec{n_1}|=\sqrt{(-60)^2+144^2+(-16)^2}=\sqrt{24592}$$

$$\vec{n_2}=\overrightarrow{SA}\times\overrightarrow{SC}=\left[\begin{vmatrix}3 & 12\\12 & 3\end{vmatrix},-\begin{vmatrix}4 & 12\\4 & 3\end{vmatrix},\begin{vmatrix}4 & 3\\4 & 12\end{vmatrix}\right]=[9-144,-(12-48),48-12]=[-135,36,36]$$

$$\Rightarrow\ |\vec{n_2}|=\sqrt{(-135)^2+36^2+36^2}=\sqrt{20817}$$

$$\cos\angle A=\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}=\frac{[-60,144,-16]\cdot[-135,36,36]}{\sqrt{24592}\cdot\sqrt{20817}}=\frac{8100+5184-576}{\sqrt{24592}\cdot\sqrt{20817}}=\quad\Rightarrow\quad \underline{\underline{\angle A\approx 55,8^\circ}}$$

$$\vec{n_3}=\overrightarrow{SB}\times\overrightarrow{SA}=-\overrightarrow{SA}\times\overrightarrow{SB}=-[-60,144,-16]=[60,-144,16]\quad\Rightarrow\quad|\vec{n_3}|=\sqrt{24592}$$

$$\vec{n_4}=\overrightarrow{SB}\times\overrightarrow{SC}=\left[\begin{vmatrix}5 & 0\\12 & 3\end{vmatrix},-\begin{vmatrix}12 & 0\\4 & 3\end{vmatrix},\begin{vmatrix}12 & 5\\4 & 12\end{vmatrix}\right]=[15-0,-(36-0),144-20]=[15,-36,124]$$

$$|\vec{n_4}|=\sqrt{15^2+(-36)^2+124^2}=\sqrt{16897}$$

$$\cos\angle B=\frac{\vec{n_3}\cdot\vec{n_4}}{|\vec{n_3}|\cdot|\vec{n_4}|}=\frac{[60,-144,16]\cdot[15,-36,124]}{\sqrt{24592}\cdot\sqrt{16897}}=\frac{900+5184+1984}{\sqrt{24592}\cdot\sqrt{16897}}\quad\Rightarrow\quad\underline{\underline{\angle B\approx 66,7^\circ}}$$

$$\vec{n_5}=\overrightarrow{SC}\times\overrightarrow{SA}=-\overrightarrow{SA}\times\overrightarrow{SC}=-[-135,36,36]=[135,-36,-36]\quad\Rightarrow\quad|\vec{n_5}|=\sqrt{20817}$$

$$\vec{n_6}=\overrightarrow{SC}\times\overrightarrow{SB}=-\overrightarrow{SB}\times\overrightarrow{SC}=-[15,-36,124]=[-15,36,-124]\quad\Rightarrow\quad|\vec{n_6}|=\sqrt{16897}$$

$$\cos\angle C=\frac{\vec{n_5}\cdot\vec{n_6}}{|\vec{n_5}|\cdot|\vec{n_6}|}=\frac{[135,-36,-36]\cdot[-15,36,-124]}{\sqrt{20817}\cdot\sqrt{16897}}=\frac{1143}{\sqrt{20817}\cdot\sqrt{16897}}\quad\Rightarrow\quad\underline{\underline{\angle C\approx 86,5^\circ}}$$

b)

$$A=\left(\frac{A+B+C}{180}-1\right)\cdot\pi r^2=\left(\frac{55,8+66,7+86,5}{180}-1\right)\cdot\pi\cdot 13^2\approx\underline{\underline{85,5}}$$

Oppgave 5.74

a)

$$T=\left(\frac{A+B+C}{180}-1\right)\cdot \pi r^2 \quad \Leftrightarrow \quad T=\left(\frac{S}{180}-1\right)\cdot \pi r^2 \quad \Leftrightarrow \quad \frac{S}{180}-1=\frac{T}{\pi r^2} \quad \Leftrightarrow$$

$$\frac{S}{180}=\frac{T}{\pi r^2}+1 \quad \Leftrightarrow \quad \underline{\underline{S=180\cdot\left(\frac{T}{\pi r^2}+1\right)}}$$

b)

Summen av vinklene gitt av formelen $S=180\cdot\left(\frac{T}{\pi r^2}+1\right)$

$$T>0 \quad \Rightarrow \quad \frac{T}{\pi r^2}>0 \quad \Rightarrow \quad \frac{T}{\pi r^2}+1>1 \quad \Rightarrow \quad S>180$$

5.8 Sfæriske koordinater

Oppgave 5.80

a) Kartesiske koordinater $(5,0,0)$ og radius 5:

$$\tan u = \frac{y}{x} = \frac{0}{5} = 0 \quad \Rightarrow \quad u = 0°$$

$$\sin v = \frac{z}{r} = \frac{0}{5} = 0 \quad \Rightarrow \quad v = 0°$$

Sfæriske koordinater $\underline{\underline{(0°, 0°)}}$.

b) Kartesiske koordinater $(0,5,0)$ og radius 5:

$$\tan u = \frac{y}{x} = \frac{5}{0} \text{ Ikke definert} \quad \Rightarrow \quad u = 90°$$

$$\sin v = \frac{z}{r} = \frac{0}{5} = 0 \quad \Rightarrow \quad v = 0°$$

Sfæriske koordinater $\underline{\underline{(90°, 0°)}}$.

c) Kartesiske koordinater $(3,4,0)$ og radius 5:

$$\tan u = \frac{y}{x} = \frac{4}{3} \quad \Rightarrow \quad u \approx 53{,}1°$$

$$\sin v = \frac{z}{r} = \frac{0}{5} = 0 \quad \Rightarrow \quad v = 0°$$

Sfæriske koordinater $\underline{\underline{(53{,}1°\ ,\ 0°)}}$.

d) Kartesiske koordinater $(0,3,4)$ og radius 5:

$$\tan u = \frac{y}{x} = \frac{3}{0} \quad \Rightarrow \quad u = 90°$$

$$\sin v = \frac{z}{r} = \frac{4}{5} = 0 \quad \Rightarrow \quad v \approx 53{,}1°$$

Sfæriske koordinater $\underline{\underline{(90°,\ 53{,}1°)}}$.

e) Kartesiske koordinater $(2,3,2\sqrt{3})$ og radius 5:

$$\tan u = \frac{y}{x} = \frac{3}{2} \quad \Rightarrow \quad u \approx 56{,}3°$$

$$\sin v = \frac{z}{r} = \frac{2\sqrt{3}}{5} \quad \Rightarrow \quad v \approx 43{,}9°$$

Sfæriske koordinater $\underline{\underline{(56{,}3°\ ,\ 43{,}9°)}}$.

Oppgave 5.81

a) Kartesiske koordinater $(9,12,0)$ og radius 15:

$$\tan u = \frac{y}{x} = \frac{12}{9} \quad \Rightarrow \quad u \approx 53{,}1^\circ$$

$$\sin v = \frac{z}{r} = \frac{0}{15} = 0 \quad \Rightarrow \quad v = 0^\circ$$

Sfæriske koordinater $\underline{\underline{\left(53{,}1^\circ\ ,\ 0^\circ\right)}}$.

b) Kartesiske koordinater $(5,10,10)$ og radius 15:

$$\tan u = \frac{y}{x} = \frac{10}{5} = 2 \quad \Rightarrow \quad u \approx 63{,}4^\circ$$

$$\sin v = \frac{z}{r} = \frac{10}{15} \quad \Rightarrow \quad v \approx 41{,}8^\circ$$

Sfæriske koordinater $\underline{\underline{\left(63{,}4^\circ\ ,\ 41{,}8^\circ\right)}}$.

c) Kartesiske koordinater $(-10,10,5)$ og radius 15:

$$\tan u = \frac{y}{x} = \frac{10}{-10} = -1 \quad \Rightarrow \quad u = 135^\circ$$

$$\sin v = \frac{z}{r} = \frac{5}{15} \quad \Rightarrow \quad v \approx 19{,}5^\circ$$

Sfæriske koordinater $\underline{\underline{\left(135^\circ, 19{,}5^\circ\right)}}$.

d) Kartesiske koordinater $(-5,10,-10)$ og radius 15:

$$\tan u = \frac{y}{x} = \frac{10}{-5} = -2 \quad \Rightarrow \quad u \approx 116{,}6^\circ$$

$$\sin v = \frac{z}{r} = \frac{-10}{15} \quad \Rightarrow \quad v \approx -41{,}8^\circ$$

Sfæriske koordinater $\underline{\underline{\left(116{,}6^\circ\ ,-41{,}8^\circ\right)}}$.

Oppgave 5.82

a) Radius 12 og sfæriske koordinater $\left(45^\circ,45^\circ\right)$

$$x = 12 \cdot \cos 45^\circ \cdot \cos 45^\circ = 12 \cdot \tfrac{\sqrt{2}}{2} \cdot \tfrac{\sqrt{2}}{2} = 12 \cdot \tfrac{2}{4} = 6$$

$$y = 12 \cdot \cos 45^\circ \cdot \sin 45^\circ = 12 \cdot \tfrac{\sqrt{2}}{2} \cdot \tfrac{\sqrt{2}}{2} = 12 \cdot \tfrac{2}{4} = 6$$

$$z = 12 \cdot \sin 45^\circ = 12 \cdot \tfrac{\sqrt{2}}{2} = 6\sqrt{2}$$

Kartesiske koordinater $\underline{\underline{\left(6, 6, 6\sqrt{2}\right)}}$.

b) Radius 12 og sfæriske koordinater $(-30°, 60°)$

$x = 12 \cdot \cos 60° \cdot \cos(-30°) = 12 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$

$y = 12 \cdot \cos 60° \cdot \sin(-30°) = 12 \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) = -3$ Kartesiske koordinater $\underline{\underline{\left(3\sqrt{3}, -3, 6\sqrt{3}\right)}}$.

$z = 12 \cdot \sin 60° = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$

c) Radius 12 og sfæriske koordinater $(22{,}3°, -78{,}0°)$

$x = 12 \cdot \cos(-78{,}0°) \cdot \cos 22{,}3° \approx 2{,}31$

$y = 12 \cdot \cos(-78{,}0°) \cdot \sin 22{,}3° \approx 0{,}95$ Kartesiske koordinater $\underline{\underline{(2{,}31\ ,\ 0{,}95\ ,\ -11{,}74)}}$.

$z = 12 \cdot \sin(-78{,}0°) \approx -11{,}74$

d) Radius 12 og sfæriske koordinater $(-15{,}3°, -44{,}5°)$

$x = 12 \cdot \cos(-44{,}5°) \cdot \cos(-15{,}3°) \approx 8{,}26$

$y = 12 \cdot \cos(-44{,}5°) \cdot \sin(-15{,}3°) \approx -2{,}26$ Kartesiske koordinater $\underline{\underline{(8{,}26\ ,\ -2{,}26\ ,\ -8{,}41)}}$.

$z = 12 \cdot \sin(-44{,}5°) \approx -8{,}41$

Oppgave 5.83

Oslo - Nordpolen: $\frac{90° - 59{,}92°}{180°} \cdot \pi \cdot 6380 \text{ km} \approx 3349{,}5 \text{ km}$

Nordpolen - Tasmania: $\frac{90° + 42{,}0°}{180°} \cdot \pi \cdot 6380 \text{ km} \approx 14698{,}5 \text{ km}$

Oslo - Nordpolen-Tasmania $\approx \underline{\underline{18048 \text{ km}}}$

Oslo - Sørpolen: $\frac{59{,}92° + 90°}{180°} \cdot \pi \cdot 6380 \text{ km} \approx 16694 \text{ km}$

Sørpolen - Tasmania: $\frac{90° - 42{,}0°}{180°} \cdot \pi \cdot 6380 \text{ km} \approx 5345 \text{ km}$

Oslo - Sørpolen-Tasmania $\approx \underline{\underline{22039 \text{ km}}}$

Oppgave 5.84

Oslo - Nordpolen: $\frac{90° - 59,92°}{180°} \cdot \pi \cdot 6380 \text{ km} \qquad \approx 3349,5 \text{ km}$

Nordpolen - Honolulu: $\frac{90° - 21,32°}{180°} \cdot \pi \cdot 6380 \text{ km} \approx 7647,5 \text{ km}$

Oslo - Nordpolen-Honolulu $\qquad \approx \underline{\underline{10997 \text{ km}}}$

Oslo - Ekvator: $\frac{59,92°}{180°} \cdot \pi \cdot 6380 \text{ km} \qquad \approx 6672 \text{ km}$

Langs Ekvator: $\frac{10,75° + 157,87°}{180°} \cdot \pi \cdot 6380 \text{ km} \approx 18776 \text{ km}$

Ekvator - Honolulu: $\frac{21,32°}{180°} \cdot \pi \cdot 6380 \text{ km} \qquad \approx 2374 \text{ km}$

Oslo - Sørpolen-Tasmania $\qquad \approx \underline{\underline{27822 \text{ km}}}$

Oppgave 5.85

a)

Kula $K: \begin{cases} x = 2 + 15\cos v \cos u \\ y = 3 + 15\cos v \sin u \\ z = -4 + 15\sin v \end{cases}$ har radius 15 og sentrum i $(2,3,-4)$.

$\Rightarrow \quad \underline{\underline{(x-2)^2 + (y-3)^2 + (z+4)^2 = 15^2}}$

b)

$P(0,13,7)$ ligger på kula hvis $(0-2)^2 + (13-3)^2 + (7+4)^2 = 15^2$

$(-2)^2 + 10^2 + 11^2 = 225 = 15^2 \qquad \underline{\underline{P(0,13,7) \text{ ligger på kula.}}}$

c) $2+15\cos v\cos u=0 \quad (1)$

$3+15\cos v\sin u=13 \quad (2)$

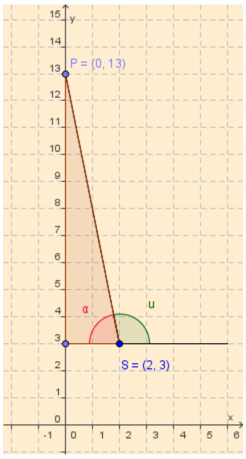
$-4+15\sin v=7 \quad (3)$

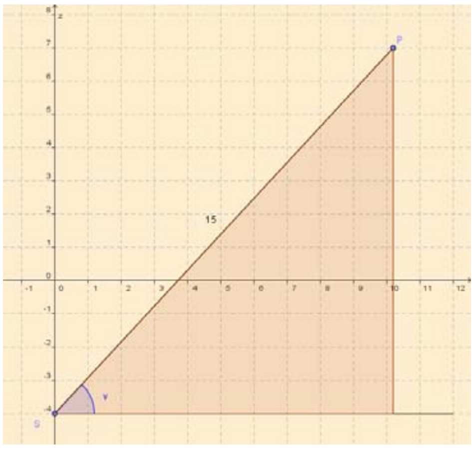
$(3)\quad 15\sin v=11 \;\Leftrightarrow\; \sin v=\frac{11}{15} \;\Rightarrow\; \underline{\underline{v\approx 47{,}2^\circ}}$

$(1)\quad 2+15\cos 47{,}2^\circ\cdot\cos u=0 \;\Leftrightarrow\; \cos u=\dfrac{-2}{15\cos 47{,}2^\circ} \;\Rightarrow\; \underline{\underline{u\approx 101{,}3^\circ}}$

Kontroll av (2) $\quad 3+15\cos 47{,}2^\circ\cdot\sin 101{,}3^\circ\approx 13$

Geometrisk løst:





$\tan\alpha=\dfrac{10}{2}=5 \;\Rightarrow\; \alpha\approx 78{,}7^\circ$

$\underline{\underline{u=180^\circ-78{,}7^\circ=101{,}3^\circ}}$

$\sin v=\dfrac{11}{15} \;\Rightarrow\; \underline{\underline{v\approx 47{,}2^\circ}}$

Oppgave 5.86

a) Gitt kula $(x-1)^2+(y-2)^2+(z+3)^2=5^2$

Sentrum i $(1,2,-3)$ og radius 5 $\;\Rightarrow\; \underline{\underline{K:\begin{cases} x=1+5\cos v\cos u \\ y=2+5\cos v\sin u \\ z=-3+5\sin v\end{cases}}}$

b) Gitt kula $x^2+y^2+z^2-6x+4y-8z+4=0 \;\Leftrightarrow$

$x^2-6x+y^2+4y+z^2-8z=-4 \;\Leftrightarrow$

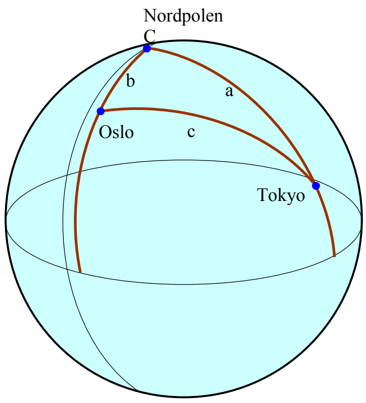
$x^2-6x+\left(\frac{6}{2}\right)^2+y^2+4y+\left(\frac{4}{2}\right)^2+z^2-8z+\left(\frac{8}{2}\right)^2=-4+\left(\frac{6}{2}\right)^2+\left(\frac{4}{2}\right)^2+\left(\frac{8}{2}\right)^2 \;\Leftrightarrow$

$(x-3)^2+(y+2)^2+(z-4)^2=25=5^2$

Sentrum i $(3,-2,4)$ og radius 5 $\;\Rightarrow\; \underline{\underline{K:\begin{cases} x=3+5\cos v\cos u \\ y=-2+5\cos v\sin u \\ z=4+5\sin v\end{cases}}}$

5.9 Sfærisk trigonometri

Oppgave 5.90



$\angle C = 139,75° - 10,75° = 129°$

$a = 90° - 35,75° = 54,25°$

$b = 90° - 59,92° = 30,08°$

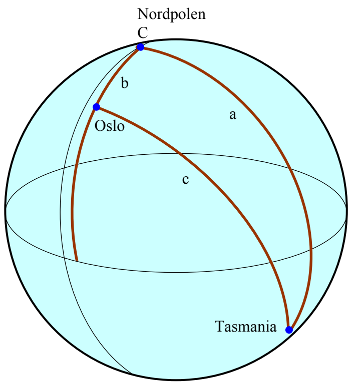
$\cos c = \cos 54,25° \cdot \cos 30,08° + \sin 54,25° \cdot \sin 30,08° \cdot \cos 129°$

$\Rightarrow \quad c \approx 75,55°$

Avstand flyet tilbakelegger på turen mellom Oslo og Tokyo:

$\frac{75,55}{180} \cdot \pi \cdot 6380 \text{ km} \approx \underline{\underline{8412 \text{ km}}}$

Oppgave 5.91



$\angle C = 146,5° - 10,75° = 135,75°$

$a = 90° + 42° = 132°$

$b = 90° - 59,92° = 30,08°$

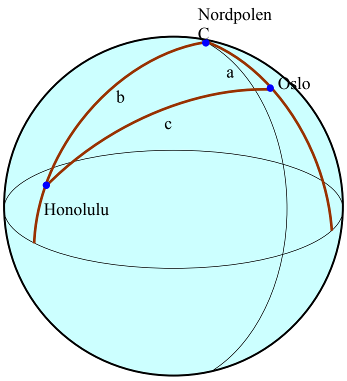
$\cos c = \cos 132° \cdot \cos 30,08° + \sin 132° \cdot \sin 30,08° \cdot \cos 135,75°$

$\Rightarrow \quad c \approx 147,8°$

Avstand flyet tilbakelegger på turen mellom Oslo og Tasmania:

$\frac{147,8}{180} \cdot \pi \cdot 6380 \text{ km} \approx \underline{\underline{16453 \text{ km}}}$

Oppgave 5.92



$\angle C = 157,87° + 10,75° = 168,62°$

$a = 90° - 59,92° = 30,08°$

$b = 90° - 21,32° = 68,68°$

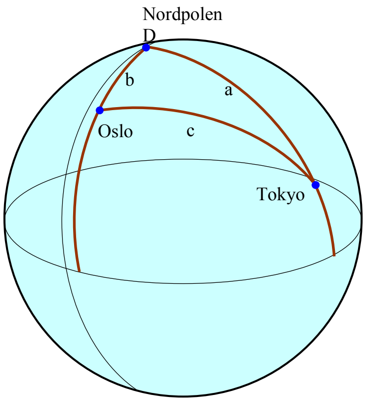
$\cos c = \cos 30,08° \cdot \cos 68,68° + \sin 30,08° \cdot \sin 68,68° \cdot \cos 168,62°$

$\Rightarrow \quad c \approx 98,2°$

Avstand flyet tilbakelegger på turen mellom Oslo og Honolulu:

$\frac{98,2}{180} \cdot \pi \cdot 6380 \text{ km} \approx \underline{\underline{10938 \text{ km}}}$

Oppgave 5.93



Fra oppgave 5.90 har vi:

$\angle C = 129°$

$a = 54,25°$

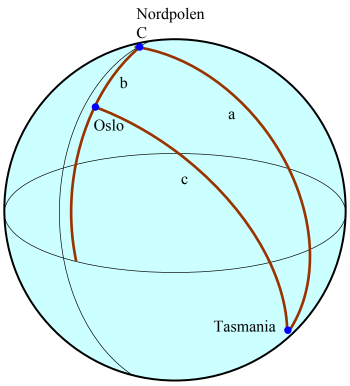
$b = 30,08°$

$c = 75,55°$

$$\frac{\sin \angle A}{\sin 54,25°} = \frac{\sin 129°}{\sin 75,55°} \quad \Rightarrow \quad \angle A \approx 40,6°$$

<u>Flyet må holde kursen $40,6°$ østenfor nord.</u>

Oppgave 5.94



Fra oppgave 5.91 har vi:

$\angle C = 135,75°$

$a = 132°$

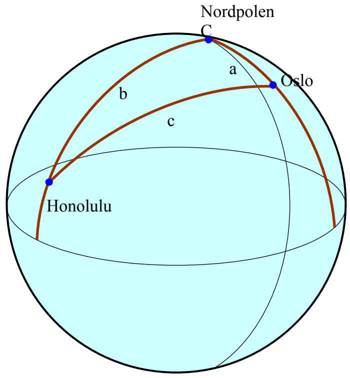
$b = 30,08°$

$c = 147,8°$

$$\frac{\sin \angle A}{\sin 132°} = \frac{\sin 135,75°}{\sin 147,8°} \quad \Rightarrow \quad \cancel{\angle A \approx 76,7°} \quad \vee \quad \angle A \approx 103,3°$$

<u>Flyet må holde kursen $103,3°$ østenfor nord eller $76,7°$ østenfor sør.</u>

Oppgave 5.95



Fra oppgave 5.92 har vi:

$\angle C = 168,62°$

$a = 30,08°$

$b = 68,68°$

$c = 98,2°$

$$\frac{\sin \angle B}{\sin 68,68°} = \frac{\sin 168,62°}{\sin 98,2°} \quad \Rightarrow \quad \angle B \approx 10,7°$$

Flyet må holde kursen 10,7° vestenfor nord.

6.1 Tallfølger

Oppgave 6.10

$$a_n = 5n - 2 \quad \Rightarrow \quad \begin{aligned} a_1 &= 5\cdot 1 - 2 = 5 - 2 = 3 \\ a_2 &= 5\cdot 2 - 2 = 10 - 2 = 8 \\ a_3 &= 5\cdot 3 - 2 = 15 - 2 = 13 \end{aligned} \qquad \begin{aligned} a_4 &= 5\cdot 4 - 2 = 20 - 2 = 18 \\ a_5 &= 5\cdot 5 - 2 = 25 - 2 = 23 \\ a_6 &= 5\cdot 6 - 2 = 30 - 2 = 28 \end{aligned}$$

De seks første leddene i følgen er 3,8,13,18,23,28.

Oppgave 6.11

$$a_n = 2\cdot n^2 \quad \Rightarrow \quad \begin{aligned} a_1 &= 2\cdot 1^2 = 2 \\ a_2 &= 2\cdot 2^2 = 8 \\ a_3 &= 2\cdot 3^2 = 18 \end{aligned} \qquad \begin{aligned} a_4 &= 2\cdot 4^2 = 32 \\ a_5 &= 2\cdot 5^2 = 50 \\ a_6 &= 2\cdot 6^2 = 72 \\ a_7 &= 2\cdot 7^2 = 98 \end{aligned}$$

De sju første leddene i følgen er 2,8,18,32,50,72,98.

Oppgave 6.12

a)

$$a_i = a_{i-1} + 4 \ \wedge \ a_1 = 3 \quad \Rightarrow \quad \begin{aligned} a_1 &= 3 \\ a_2 &= 3 + 4 = 7 \\ a_3 &= 7 + 4 = 11 \end{aligned} \qquad \begin{aligned} a_4 &= 11 + 4 = 15 \\ a_5 &= 15 + 4 = 19 \end{aligned}$$

De fem første leddene i følgen er 3,7,11,15,19.

b)

$$a_1 = 3$$
$$a_2 = 3 + 4$$
$$a_3 = 3 + 4 + 4$$
$$a_4 = 3 + 4 + 4 + 4$$
.
.
$$a_n = 3 + \underbrace{4 + .. + 4}_{(n-1)\text{ ledd}} = 3 + (n-1)\cdot 4 = 3 + 4n - 4 = \underline{\underline{4n - 1}}$$

Oppgave 6.13

a)

$$a_i = \tfrac{1}{2}a_{i-1} \;\wedge\; a_1 = 16 \quad\Rightarrow\quad \begin{aligned} a_1 &= 16 \\ a_2 &= \tfrac{1}{2}\cdot 16 = 8 \\ a_3 &= \tfrac{1}{2}\cdot 8 = 4 \end{aligned} \qquad \begin{aligned} a_4 &= \tfrac{1}{2}\cdot 4 = 2 \\ a_5 &= \tfrac{1}{2}\cdot 2 = 1 \end{aligned}$$

De fem første leddene i følgen er 16,8,4,2,1.

b)

$a_1 = 16$

$a_2 = \frac{1}{2}\cdot 16$

$a_3 = \frac{1}{2}\cdot\frac{1}{2}\cdot 16$

$a_4 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 16$

.

.

$$a_n = \underbrace{\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\ldots\cdot\tfrac{1}{2}}_{(n-1)\text{ faktorer}} = \left(\tfrac{1}{2}\right)^{n-1}\cdot 16 = \frac{1^{n-1}}{2^{n-1}}\cdot 2^4 = \frac{1}{2^{n-1}}\cdot 2^4 = 2^{4-(n-1)} = 2^{4-n+1} = \underline{\underline{2^{5-n}}}$$

6.2 Aritmetiske følger

Oppgave 6.20

a) $a_1 = -12$

$a_2 = -12 + 5 = -7$

$a_3 = -7 + 5 = -2$

$a_4 = -2 + 5 = 3$

$a_5 = 3 + 5 = 8$ De fem første leddene er $\underline{\underline{-12, -7, -2, 3, 8.}}$

b) $a_1 = 24$

$a_2 = 24 + (-2) = 22$

$a_3 = 22 + (-2) = 20$

$a_4 = 20 + (-2) = 18$

$a_5 = 18 + (-2) = 16$ De fem første leddene er $\underline{\underline{24, 22, 20, 18, 16.}}$

Oppgave 6.21

a) $5, 11, 17, 23, \ldots \quad \Rightarrow \quad \underline{\underline{d = 6}}$ og $a_n = 5 + (n-1) \cdot 6 = 5 + 6n - 6 = \underline{\underline{6n - 1}}$

b) $81, 64, 47, 30, \ldots \quad \Rightarrow \quad \underline{\underline{d = -17}}$ og $a_n = 81 + (n-1) \cdot (-17) = 81 - 17n + 17 = \underline{\underline{98 - 17n}}$

Oppgave 6.22

a) $a_1 = 100 \ \wedge \ d = 2 \quad \Rightarrow \quad a_n = 100 + (n-1) \cdot 2 = 100 + 2n - 2 = 2n + 98$

$\underline{\underline{\text{Beløpet i uke } n \text{ er gitt ved } a_n = 2n + 98}}$

b) Ukepenger om to år: $a_{104} = 2 \cdot 104 + 98 = \underline{\underline{306 \text{ kr}}}$

Oppgave 6.23

a) $a_5 = 13 \ \wedge \ d = 4$

$a_5 = a_1 + 4d \quad \Leftrightarrow \quad a_1 = a_5 - 4d \quad \Rightarrow \quad a_1 = 13 - 4 \cdot 4 = 13 - 16 = \underline{\underline{-3}}$

b) $a_n = a_1 + (n-1) \cdot d \quad \Rightarrow \quad a_n = -3 + (n-1) \cdot 4 = -3 + 4n - 4 = \underline{\underline{4n - 7}}$

c) $$a_n = 105 \;\Rightarrow\; 4n - 7 = 105 \;\Leftrightarrow\; 4n = 105 + 7 \;\Leftrightarrow\; n = \frac{112}{4} = 28$$

Leddet 105 er nummer 28 i tallfølgen.

Oppgave 6.24

a) $a_3 = 11 \;\wedge\; a_5 = 8$

$$a_5 = a_3 + 2d \;\Leftrightarrow\; d = \frac{a_5 - a_3}{2} \;\Rightarrow\; d = \frac{8 - 11}{2} = \underline{\underline{-\frac{3}{2}}}$$

$$a_3 = a_1 + 2d \;\Leftrightarrow\; a_1 = a_3 - 2d \;\Rightarrow\; a_1 = 11 - 2\cdot\left(-\tfrac{3}{2}\right) = 11 + 3 = \underline{\underline{14}}$$

b) $$a_n = 14 + (n-1)\cdot\left(-\frac{3}{2}\right) = 14 - \frac{3}{2}n + \frac{3}{2} = \underline{\underline{\frac{31}{2} - \frac{3}{2}n}}$$

6.3 Geometriske følger

Oppgave 6.30

a) $a_1 = 5 \quad \wedge \quad k = 2 \quad \Rightarrow$

$a_2 = 5 \cdot 2 = 10$

$a_3 = 10 \cdot 2 = 20$

$a_4 = 20 \cdot 2 = 40$

$a_5 = 40 \cdot 2 = 80$ De fem første leddene i tallfølgen er 5,10,20,40,80.

b) $a_1 = 16 \quad \wedge \quad k = \frac{1}{2} \quad \Rightarrow$

$a_2 = 16 \cdot \frac{1}{2} = 8$

$a_3 = 8 \cdot \frac{1}{2} = 4$

$a_4 = 4 \cdot \frac{1}{2} = 2$

$a_5 = 2 \cdot \frac{1}{2} = 1$ De fem første leddene i tallfølgen er 16,8,4,2,1.

c) $a_1 = 81 \quad \wedge \quad k = -\frac{2}{3} \quad \Rightarrow$

$a_2 = 81 \cdot \left(-\frac{2}{3}\right) = -54$

$a_3 = -54 \cdot \left(-\frac{2}{3}\right) = 36$

$a_4 = 36 \cdot \left(-\frac{2}{3}\right) = -24$

$a_5 = -24 \cdot \left(-\frac{2}{3}\right) = 16$ De fem første leddene i tallfølgen er $81, -54, 36, -24, 16$.

Oppgave 6.31

a) Gitt den geometriske tallfølgen: 1,3,9,27,...

$$k = \frac{a_i}{a_{i-}} = \frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \underline{\underline{3}}$$

b) Gitt den geometriske tallfølgen: $625, -125, 25, -5, \ldots$

$$k = \frac{a_i}{a_{i-}} = \frac{-125}{625} = \frac{25}{-125} = \frac{-5}{25} = \underline{\underline{-\frac{1}{5}}}$$

c) Gitt den geometriske tallfølgen: $\frac{2}{3}, 1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \ldots$

$$k = \frac{a_i}{a_{i-1}} = \frac{1}{\frac{2}{3}} = \frac{\frac{3}{2}}{1} = \frac{\frac{9}{4}}{\frac{3}{2}} = \frac{\frac{27}{8}}{\frac{9}{4}} = \underline{\underline{\frac{3}{2}}}$$

Oppgave 6.32

Gitt den geometriske tallfølgen: $1,3,9,27,\ldots \quad \Rightarrow \quad k=3$

$$a_n = 1\cdot 3^{n-1} = \underline{\underline{3^{n-1}}} \qquad a_{10} = 3^{10-1} = 3^9 = \underline{\underline{19\,683}}$$

Gitt den geometriske tallfølgen: $625,-125,25,-5,\ldots \quad \Rightarrow \quad k=-\frac{1}{5}$

$$\underline{\underline{a_n = 625\cdot\left(-\frac{1}{5}\right)^{n-1}}} \qquad a_{10} = 625\cdot\left(-\frac{1}{5}\right)^9 = 625\cdot\left(-\frac{1}{1953125}\right) = \underline{\underline{-\frac{1}{3125}}}$$

Gitt den geometriske tallfølgen: $\frac{2}{3},1,\frac{3}{2},\frac{9}{4},\frac{27}{8},\ldots \quad \Rightarrow \quad k=\frac{3}{2}$

$$\underline{\underline{a_n = \frac{2}{3}\cdot\left(\frac{3}{2}\right)^{n-1}}} \qquad a_{10} = \frac{2}{3}\cdot\left(\frac{3}{2}\right)^9 = \frac{2}{3}\cdot\frac{19683}{512} = \underline{\underline{\frac{6561}{256}}}$$

Oppgave 6.33

a) Gitt tallfølgen: $9,-6,4,-\frac{8}{3},\frac{16}{9}$

$$\frac{a_2}{a_1} = \frac{-6}{9} = -\frac{2}{3} \qquad \frac{a_3}{a_2} = \frac{4}{-6} = -\frac{2}{3} \qquad \frac{a_4}{a_3} = \frac{-\frac{8}{3}}{4} = -\frac{2}{3} \qquad \frac{a_5}{a_4} = \frac{\frac{16}{9}}{-\frac{8}{3}} = -\frac{2}{3} \quad \Rightarrow \quad \frac{a_i}{a_{i-1}} = \text{konstant}$$

Tallfølgen er geometrisk med kvotient $k=-\frac{2}{3}$.

b) Gitt tallfølgen: $12,9,6,4,3$

$$\frac{a_2}{a_1} = \frac{9}{12} = \frac{3}{4} \qquad \frac{a_3}{a_2} = \frac{6}{9} = \frac{2}{3} \qquad \frac{a_4}{a_3} = \frac{4}{6} = \frac{2}{3} \qquad \frac{a_5}{a_4} = \frac{3}{4} \quad \Rightarrow \quad \frac{a_i}{a_{i-1}} \neq \text{konstant}$$

Tallfølgen er ikke geometrisk.

c) Gitt tallfølgen: $1, \sqrt{2}, 2, 2\sqrt{2}, 4$

$$\frac{a_2}{a_1} = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \frac{a_3}{a_2} = \frac{2}{\sqrt{2}} = \frac{2\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2} \qquad \frac{a_4}{a_3} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\frac{a_5}{a_4} = \frac{4}{2\sqrt{2}} = \frac{4\cdot\sqrt{2}}{2\sqrt{2}\cdot\sqrt{2}} = \frac{4\sqrt{2}}{4} = \sqrt{2} \qquad \Rightarrow \qquad \frac{a_i}{a_{i-1}} = \text{konstant}$$

Tallfølgen er geometrisk med kvotient $k = \sqrt{2}$.

Oppgave 6.34

a) 1% rente per år $\Rightarrow \quad k = 1{,}01$

Saldo i 2010: $a_{1980} = 1 \cdot 1{,}01^{1980} = 360014580{,}2 \approx 360 \text{ millioner}$

b) 2% rente per år $\Rightarrow \quad k = 1{,}02$

Saldo i 2010: $a_{1980} = 1 \cdot 1{,}02^{1980} \approx 1{,}07 \cdot 10^{17}$

6.4 Rekker

Oppgave 6.40

Gitt rekka: $2+3+5+7+11+13+17+19$

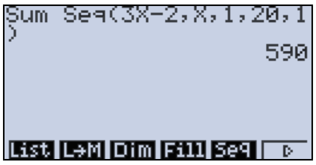
$$s_5 = 2+3+5+7+11 = \underline{\underline{28}}$$
$$s_6 = 2+3+5+7+11+13 = \underline{\underline{41}}$$
$$s_7 = 2+3+5+7+11+13+17 = \underline{\underline{58}}$$
$$s_8 = 2+3+5+7+11+13+17+19 = \underline{\underline{77}}$$

Oppgave 6.41

a) $a_n = 3n-2$

$$\left.\begin{aligned} a_1 &= 3\cdot 1-2 = 1 \\ a_2 &= 3\cdot 2-2 = 4 \\ a_3 &= 3\cdot 3-2 = 7 \\ a_4 &= 3\cdot 4-2 = 10 \\ a_5 &= 3\cdot 5-2 = 13 \\ a_6 &= 3\cdot 6-2 = 16 \end{aligned}\right\} \quad s_6 = 1+4+7+10+13+16 = \underline{\underline{51}}$$

b)



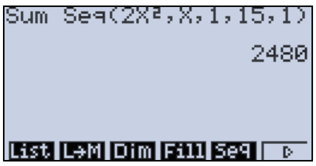
$$\underline{\underline{s_{20} = 590}}$$

Oppgave 6.42

a) $a_n = 2n^2$

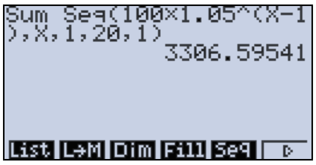
$$\left.\begin{aligned} a_1 &= 2\cdot 1^2 = 2 \\ a_2 &= 2\cdot 2^2 = 8 \\ a_3 &= 2\cdot 3^2 = 18 \\ a_4 &= 2\cdot 4^2 = 32 \\ a_5 &= 2\cdot 5^2 = 50 \end{aligned}\right\} \quad s_5 = 2+8+18+32+50 = \underline{\underline{110}}$$

b)



$$\underline{\underline{s_{15} = 2480}}$$

Oppgave 6.43



$$\sum_{i=1}^{20}\left(100\cdot 1{,}05^{i-1}\right)\approx \underline{\underline{3306{,}6}}$$

Oppgave 6.44

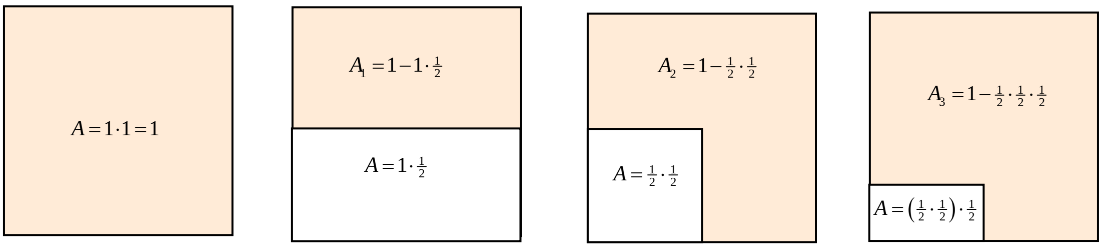
a) Gitt rekka: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

$s_1=\underline{\underline{\frac{1}{2}}}$ $\quad s_2=\frac{1}{2}+\frac{1}{4}=\underline{\underline{\frac{3}{4}}}$ $\quad s_3=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\underline{\underline{\frac{7}{8}}}$ $\quad s_4=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\underline{\underline{\frac{15}{16}}}$

b) Telleren i s_n er en mindre enn nevneren og nevneren er lik 2^n

$$\Rightarrow \quad s_n=\frac{2^n-1}{2^n}=\frac{2^n}{2^n}-\frac{1}{2^n}=\underline{\underline{1-\frac{1}{2^n}}}$$

c)



$s_1=A_1=1-\frac{1}{2}$ $\quad s_2=A_2=1-\left(\frac{1}{2}\right)^2$ $\quad s_3=A_3=1-\left(\frac{1}{2}\right)^3$

$$\Rightarrow \quad s_n=1-\left(\frac{1}{2}\right)^n=1-\frac{1^n}{2^n}=\underline{\underline{1-\frac{1}{2^n}}}$$

6.5 Aritmetiske rekker

Oppgave 6.50

a) Gitt $a_1 = 1$, $d = 5$ og $n = 10$

$$a_{10} = a_1 + (10-1)\cdot d = 1 + 9\cdot 5 = 1 + 45 = 46 \qquad s_{10} = \frac{10\cdot(1+46)}{2} = \underline{\underline{235}}$$

b) Gitt $a_1 = 100$, $d = -3$ og $n = 30$

$$a_{30} = a_1 + (30-1)\cdot d = 100 + 29\cdot(-3) = 100 - 87 = 13 \qquad s_{30} = \frac{30\cdot(100+13)}{2} = \underline{\underline{1695}}$$

c) Gitt $a_1 = 15$, $a_2 = 20$ og $n = 12$

$$d = a_2 - a_1 = 20 - 15 = 5 \qquad a_{12} = 15 + 11\cdot 5 = 15 + 55 = 70 \qquad s_{12} = \frac{12\cdot(15+70)}{2} = \underline{\underline{510}}$$

d) Gitt $a_1 = 50$, $a_{10} = 32$ og $n = 50$

$$d = \frac{a_{10} - a_1}{9} = \frac{32-50}{9} = -2 \qquad a_{50} = 50 + 49\cdot(-2) = 50 - 98 = -48 \qquad s_{50} = \frac{50\cdot(50+(-48))}{2} = \underline{\underline{50}}$$

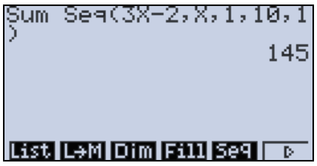
Oppgave 6.51

a) Gitt den aritmetiske rekka: $1 + 4 + 7 + \ldots + 28$

$\Rightarrow \quad a_1 = 1 \;\wedge\; d = 3$

$$1 + (n-1)\cdot 3 = 28 \;\Leftrightarrow\; 1 + 3n - 3 = 28 \;\Leftrightarrow\; 3n = 30 \;\Leftrightarrow\; n = 10$$

$$s_{10} = \frac{10\cdot(1+28)}{2} = \underline{\underline{145}}$$



$$a_n = 1 + (n-1)\cdot 3 = 1 + 3n - 3 = 3n - 2$$

b) Gitt den aritmetiske rekka: $100+98+96+...+50$

$\Rightarrow \quad a_1 = 100 \quad \wedge \quad d = -2$

$100+(n-1)\cdot(-2)=50 \quad \Leftrightarrow \quad 100-2n+2=50 \quad \Leftrightarrow$

$2n=52 \quad \Leftrightarrow \quad n=26 \qquad s_{26}=\dfrac{26\cdot(100+50)}{2}=\underline{\underline{1950}}$

$a_n = 100+(n-1)\cdot(-2)=100-2n+2=102-2n$

c) Gitt den aritmetiske rekka: $1+2+3+4+...+1000$

$\Rightarrow \quad a_1 = 1 \quad \wedge \quad d=1 \quad \wedge \quad n=1000$

$s_{1000}=\dfrac{1000\cdot(1+1000)}{2}=\underline{\underline{500500}}$

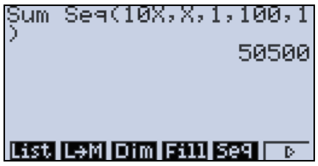
Antall ledd (*n*=1000) er for stort for lommeregneren.

$a_n = 1+(n-1)\cdot 1 = 1+n-1=n$

d) Gitt den aritmetiske rekka: $10+20+30+...+1000$

$\Rightarrow \quad a_1 = 10 \quad \wedge \quad d=10$

$10+(n-1)\cdot 10=1000 \quad \Leftrightarrow \quad 10+10n-10=1000 \quad \Leftrightarrow$



$10n=1000 \quad \Leftrightarrow \quad n=100 \qquad s_{100}=\dfrac{100\cdot(10+1000)}{2}=\underline{\underline{50500}}$

$a_n = 10+(n-1)\cdot 10=10+10n-10=10n$

Oppgave 6.52

a) Gitt den aritmetiske rekka: $1+2+3+...+9999$

$\Rightarrow \quad a_1 = 1 \quad \wedge \quad d=1 \quad \wedge \quad n=9999$

Summen av de naturlige tallene som er mindre enn 10 000: $\dfrac{9999\cdot(1+9999)}{2}=\underline{\underline{49\,995\,000}}$

b) Gitt den aritmetiske rekka: $1+3+5+...+9999$

$\Rightarrow \quad a_1 = 1 \;\wedge\; d = 2$

$$9999 = 1+(n-1)\cdot 2 \quad \Leftrightarrow \quad 9999 = 1+2n-2 \quad \Leftrightarrow \quad 2n = 10000 \quad \Leftrightarrow \quad n = 5000$$

Summen av oddetallene som er mindre enn 10 000: $\dfrac{5000\cdot(1+9999)}{2} = \underline{\underline{25\,000\,000}}$

c) Summen av alle partall mindre enn 10000 =

Summen av alle naturlige tall mindre enn 10000 $-$ Summen av alle oddetall mindre enn 10000

$\Rightarrow$ Summen av alle partall mindre enn 10000 $= 49\,995\,000 - 25\,000\,000 = \underline{\underline{24\,995\,000}}$

Oppgave 6.53

Gitt den aritmetiske rekka: $1+3+5+7+...$

$$a_1 = 1 \;\wedge\; d = 2 \quad \Rightarrow \quad a_n = 1+(n-1)\cdot 2 = 1+2n-2 = 2n-1$$

$$s_n = \frac{n\cdot(1+(2n-1))}{2} = \frac{n\cdot \cancel{2}n}{\cancel{2}} = \underline{\underline{n^2}}$$

Oppgave 6.54

Gitt $a_1 = 1 \;\wedge\; d = 7$

$$a_n = 1+(n-1)\cdot 7 = 1+7n-7 = 7n-6$$

$$s_n = 1350 \quad \Rightarrow \quad \frac{n\cdot(1+(7n-6))}{2} = 1350 \quad \Leftrightarrow \quad n\cdot(7n-5) = 2700 \quad \Leftrightarrow \quad 7n^2-5n-2700 = 0 \quad \Leftrightarrow$$

$$n = \frac{-(-5)\pm\sqrt{(-5)^2-4\cdot 7\cdot(-2700)}}{2\cdot 7} = \frac{5\pm\sqrt{75625}}{14} = \frac{5\pm 275}{14} \quad \Leftrightarrow \quad n = 20 \;\vee\; \cancel{n \approx -19{,}3}$$

<u>Det er 20 ledd i rekka.</u>

Oppgave 6.55

Gitt $a_1 = 100 \;\wedge\; d = 2$

$$a_n = 100 + (n-1)\cdot 2 = 100 + 2n - 2 = 2n + 98$$

$$s_n = 10000 \;\Rightarrow\; \frac{n\cdot(100+(2n+98))}{2} = 10000 \;\Leftrightarrow\; n\cdot(2n+198) = 20000 \;\Leftrightarrow$$

$$2n^2 + 198n - 20000 = 0 \;\Leftrightarrow\; n = \frac{-198 \pm \sqrt{198^2 - 4\cdot 2\cdot(-20000)}}{2\cdot 2} = \frac{-198 \pm \sqrt{199204}}{4} \approx \frac{-198 \pm 446,3}{4} \;\Leftrightarrow$$

$$n \approx 62,1 \;\vee\; \cancel{n \approx -161,1}$$

Etter 63 uker har hun passert 10 000 kroner utbetalt.

Oppgave 6.56

a) Gitt $a_1 = 200\text{mill kr} \;\wedge\; d = 15\text{mill kr}$

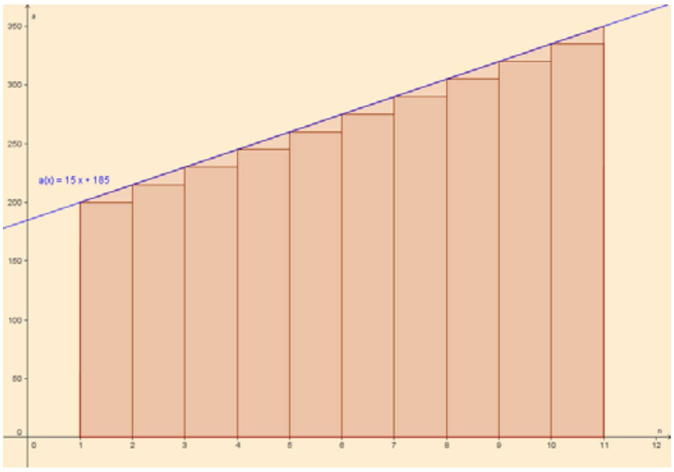
$$\Rightarrow\; a_n = 200 + (n-1)\cdot 15 = 200 + 15n - 15 = 15n + 185$$

$$a_{10} = 15\cdot 10 + 185 = 335$$

$$s_{10} = \frac{10\cdot(200+335)}{2} = 2675$$ Samlet omsetning i perioden 2008 – 2017 blir 2675 millioner kroner.

b) $\int_1^{11} (15n + 185)\,dn = 2750$ Samlet omsetning blir 2750 millioner kroner.

c)



Når samlet omsetning finnes ved å summere en rekke, tilsvarer dette totalarealet av de ti brune rektanglene på figuren til venstre.
Ved å bruke integrasjon vil svaret bli for stort fordi man da også får med arealene av de små, lysebrune trekantene på toppen av hvert rektangel.

Oppgave 6.57

$$s_n = \frac{n \cdot (a_1 + a_n)}{2} = \frac{n \cdot \left(a_1 + \left(a_1 + (n-1) \cdot d\right)\right)}{2} = \frac{n \cdot \left(a_1 + a_1 + (n-1) \cdot d\right)}{2} = \frac{n \cdot \left(2a_1 + (n-1) \cdot d\right)}{2} \quad \Leftrightarrow$$

$$s_n = \frac{n \cdot \cancel{2} a_1}{\cancel{2}} + \frac{n \cdot (n-1) \cdot d}{2} = n \cdot a_1 + \frac{n \cdot (n-1)}{2} \cdot d \quad \Leftrightarrow \quad \underline{\underline{s_n = n \cdot a_1 + \frac{n \cdot (n-1)}{2} \cdot d}}$$

6.6 Geometriske rekker

Oppgave 6.60

a) Gitt en geometrisk rekke med $a_1 = 1$, $k = 2$ og $n = 10$

$$s_{10} = 1 \cdot \frac{2^{10} - 1}{2 - 1} = 1 \cdot \frac{1024 - 1}{1} = \underline{\underline{1023}}$$

b) Gitt en geometrisk rekke med $a_1 = 3$, $k = -\frac{1}{2}$ og $n = 10$

$$s_{10} = 3 \cdot \frac{\left(-\frac{1}{2}\right)^{10} - 1}{-\frac{1}{2} - 1} = 3 \cdot \frac{\frac{1}{1024} - 1}{-\frac{3}{2}} = \underline{\underline{\frac{1023}{512}}} \approx \underline{\underline{1,998}}$$

c) Gitt en geometrisk rekke med $a_1 = 10$, $a_2 = 12$ og $n = 15$ $\Rightarrow k = \frac{a_2}{a_1} = \frac{12}{10} = 1,2$

$$s_{15} = 10 \cdot \frac{1,2^{15} - 1}{1,2 - 1} \approx 10 \cdot \frac{14,41}{0,2} \approx \underline{\underline{720,4}}$$

Oppgave 6.61

a) Gitt den geometrisk rekken $1 + 3 + 9 + 27 + 81 + 243$ $\Rightarrow$ $a_1 = 1$, $k = \frac{3}{1} = 3$ og $n = 6$

$$s_6 = 1 \cdot \frac{3^6 - 1}{3 - 1} = 1 \cdot \frac{729 - 1}{2} = \underline{\underline{364}}$$

b) Gitt den geometrisk rekken $384 - 192 + 96 - 48 + 24 - 12$ $\Rightarrow$ $a_1 = 384$, $k = \frac{-192}{384} = -\frac{1}{2}$ og $n = 6$

$$s_6 = 384 \cdot \frac{\left(-\frac{1}{2}\right)^6 - 1}{-\frac{1}{2} - 1} = 384 \cdot \frac{\frac{1}{64} - 1}{-\frac{3}{2}} = \underline{\underline{252}}$$

c) Gitt den geometrisk rekken $100 + 120 + 144 + 172,8 + 207,36$ $\Rightarrow$ $a_1 = 100$, $k = \frac{120}{100} = 1,2$ og $n = 5$

$$s_6 = 100 \cdot \frac{1,2^5 - 1}{1,2 - 1} = \underline{\underline{744,16}}$$

d) Gitt den geometrisk rekken $5+10+20+...+640 \quad \Rightarrow \quad a_1 = 5 \;,\; k = \frac{10}{5} = 2$

$a_n = 640 \quad \Rightarrow \quad 5 \cdot 2^{n-1} = 640 \quad \Leftrightarrow \quad 2^{n-1} = \frac{640}{5} \quad \Leftrightarrow \quad 2^{n-1} = 128 \quad \Leftrightarrow \quad 2^{n-1} = 2^7 \quad \Leftrightarrow$
$n-1 = 7 \quad \Leftrightarrow \quad n = 8$

$$s_8 = 5 \cdot \frac{2^8 - 1}{2-1} = 5 \cdot \frac{256-1}{1} = \underline{\underline{1275}}$$

e) Gitt den geometrisk rekken $50 + 50 \cdot 1,05 + ... + 50 \cdot 1,05^{19} \quad \Rightarrow \quad a_1 = 50 \;,\; k = 1,05$ og $n = 20$

$$s_{20} = 50 \cdot \frac{1,05^{20} - 1}{1,05 - 1} \approx \underline{\underline{1653,30}}$$

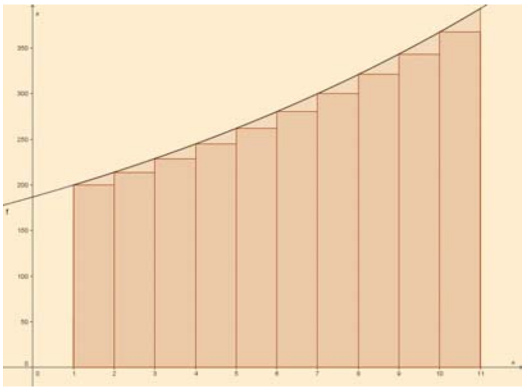
Oppgave 6.62

a) Omsetningen blir en geometrisk rekke med $a_1 = 200$ mill. kr og $k = 1,07$.

Omsetningen om ti år: $200 \text{ mill. kr} \cdot 1,07^{10} \approx \underline{\underline{393,4 \text{ mill. kr}}}$

b) Samlet omsetning i tiårsperioden: $200 \text{ mill. kr} \cdot \frac{1,07^{10} - 1}{1,07 - 1} \approx \underline{\underline{2763 \text{ mill. kr}}}$

c)



Samlet omsetning i tiårsperioden:

$$\int_1^{11} 200 \cdot 1,07^{n-1} dn = \left[200 \cdot \frac{1}{\ln 1,07} 1,07^{n-1} \right]_1^{11} \approx \underline{\underline{2859 \text{ mill kr}}}$$

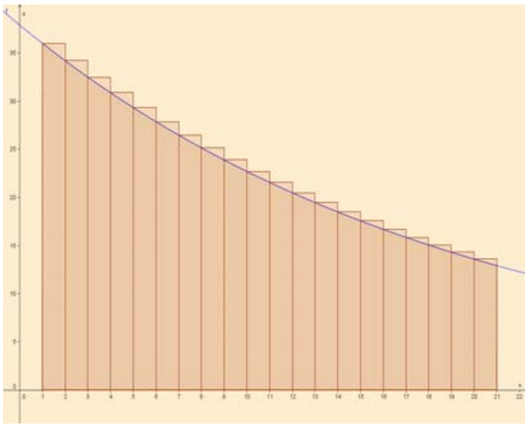
Oppgave 6.63

a) Utslippet blir en geometrisk rekke med $a_1 = 36 \text{ tonn}$ og $k = 0,95$.

Utslipp om 20 år: $36 \text{ tonn} \cdot 0,95^{20} \approx \underline{\underline{12,9 \text{ tonn}}}$

b) Samlet utslipp i tyveårsperioden: $36 \text{ tonn} \cdot \frac{0,95^{20} - 1}{0,95 - 1} \approx \underline{\underline{462 \text{ tonn}}}$

c)



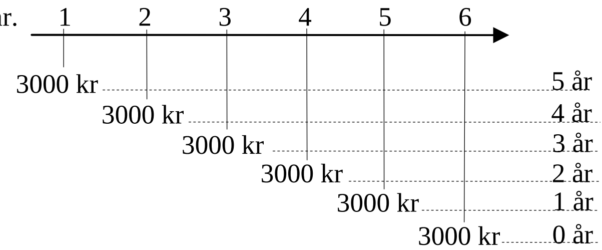
Samlet utslipp i tyveårsperioden:

$$\int_1^{21} 36\cdot 0,95^{n-1}dn = \left[36\cdot\frac{1}{\ln 0,95}0,95^{n-1}\right]_1^{21} \approx \underline{\underline{450\,\text{tonn}}}$$

d) Ved integrasjon får vi litt for lavt svar fordi vi mister arealene av de små, lysebrune trekantene over kurva.

Oppgave 6.64

a)



b) Innskuddene blir en geometrisk rekke med $a_1 = 3000$ og $k = 1,05$.

$$s_6 = 3000\cdot\frac{1,05^6 - 1}{1,05 - 1} \approx 20406$$

<u>Otto kan ta ut 20 406 kr.</u>

Oppgave 6.65

a) Innskuddene blir en geometrisk rekke med $a_1 = 5000$, $k = 1,04$ og 11 ledd.

$$s_{11} = 5000\cdot\frac{1,04^{11} - 1}{1,04 - 1} \approx 67431,76$$

<u>Mari hadde da 67431,76 kr i banken.</u>

b)

$$s_n = 500\,000 \;\Rightarrow\; 5000\cdot\frac{1,04^n - 1}{1,04 - 1} = 500000 \;\Leftrightarrow\; \frac{1,04^n - 1}{0,04} = 100 \;\Leftrightarrow\; 1,04^n - 1 = 4 \;\Leftrightarrow$$

$$1,04^n = 5 \;\Leftrightarrow\; \lg 1,04^n = \lg 5 \;\Leftrightarrow\; n\cdot\lg 1,04 = \lg 5 \;\Leftrightarrow\; n = \frac{\lg 5}{\lg 1,04} \approx 41,04$$

500 000 kr står på kontoen etter litt mer enn 41 innskudd, dvs. om litt mer enn 40 år.

<u>Etter 41 år vil Mari ha 500 000 kr i banken.</u>

Oppgave 6.66

Innskuddene blir en geometrisk rekke med $k = 1,05$ og $a_1 = 4000 \cdot 1,05$.

$s_{40} = 4000 \cdot 1,05 \cdot \dfrac{1,05^{40} - 1}{1,05 - 1} \approx 507359$ Ole har da 507 359 kr i banken.

6.7 Uendelige rekker

Oppgave 6.70

a) Gitt den uendelige rekka: $625+125+25+5+...$ Geometrisk rekke med kvotient $k=\frac{125}{625}=\frac{1}{5}$

$$\Rightarrow \quad s_n = 625\cdot\frac{\left(\frac{1}{5}\right)^n-1}{\frac{1}{5}-1} = 625\cdot\frac{\left(\frac{1}{5}\right)^n-1}{-\frac{4}{5}} = -\frac{3125}{4}\cdot\left(\left(\frac{1}{5}\right)^n-1\right) = \underline{\underline{\frac{3125}{4}\cdot\left(1-\left(\frac{1}{5}\right)^n\right)}}$$

$n\to\infty \quad\Rightarrow\quad \left(\frac{1}{5}\right)^n\to 0 \quad\Rightarrow\quad s_n\to\frac{3125}{4}$ <u>Rekka konvergerer og har summen $\frac{3125}{4}$.</u>

b) Gitt den uendelige rekka: $2+5+8+11+...$ Aritmetisk rekke med $a_1=2 \wedge d=3$.

$$\Rightarrow \quad s_n = n\cdot\frac{2+\left(2+(n-1)\cdot 3\right)}{2} = n\cdot\frac{2+2+3n-3}{2} = \underline{\underline{\frac{n\cdot(3n+1)}{2}}}$$

$n\to\infty \quad\Rightarrow\quad s_n\to\infty$ <u>Rekka divergerer.</u>

c) Gitt den uendelige rekka: $100+100\cdot 1{,}1+100\cdot 1{,}1^2+...$ Geometrisk rekke med kvotient $k=1{,}1$.

$$\Rightarrow \quad s_n = 100\cdot\frac{1{,}1^n-1}{1{,}1-1} = 100\cdot\frac{1{,}1^n-1}{0{,}1} = \underline{\underline{1000\cdot\left(1{,}1^n-1\right)}}$$

$n\to\infty \quad\Rightarrow\quad 1{,}1^n\to\infty \quad\Rightarrow\quad s_n\to\infty$ <u>Rekka divergerer.</u>

d) Gitt den uendelige rekka: $100+100\cdot 0{,}9+100\cdot 0{,}9^2+...$ Geometrisk rekke med kvotient $k=0{,}9$.

$$\Rightarrow \quad s_n = 100\cdot\frac{0{,}9^n-1}{0{,}9-1} = 100\cdot\frac{0{,}9^n-1}{-0{,}1} = -1000\cdot\left(0{,}9^n-1\right) = \underline{\underline{1000\cdot\left(1-0{,}9^n\right)}}$$

$n\to\infty \quad\Rightarrow\quad 0{,}9^n\to 0 \quad\Rightarrow\quad s_n\to 1000$ <u>Rekka konvergerer og har summen 1000.</u>

Oppgave 6.71

a) Gitt den uendelige rekka: $100+50+25+...$ Geometrisk rekke med kvotient $k=\frac{50}{100}=\frac{1}{2}$.

$k\in\langle -1,1\rangle \Rightarrow$ Rekka konvergerer. Summen blir: $s=\frac{100}{1-\frac{1}{2}}=\underline{\underline{200}}$

b) Gitt den uendelige rekka: $1+1,5+2,25+...$ Geometrisk rekke med kvotient $k=\frac{1,5}{1}=1,5$.

$k\notin\langle -1,1\rangle \Rightarrow$ Rekka divergerer.

c) Gitt den uendelige rekka: $10-9+8,1-7,29+...$ Geometrisk rekke med kvotient $k=\frac{-9}{10}=-\frac{9}{10}$.

$k\in\langle -1,1\rangle \Rightarrow$ Rekka konvergerer. Summen blir: $s=\frac{10}{1-(-\frac{9}{10})}=\frac{10}{\frac{19}{10}}=\underline{\underline{\frac{100}{19}}}$

d) Gitt den uendelige rekka: $10-11+12,1-13,31+...$ Geometrisk rekke med kvotient $k=\frac{-11}{10}=-1,1$.

$k\notin\langle -1,1\rangle \Rightarrow$ Rekka divergerer.

Oppgave 6.72

a) Beløpene blir ei geometrisk rekke med $a_1=1000 \wedge k=0,99$.

$a_{12}=1000\cdot 0,99^{11}\approx 895$ Det 12.beløpet er 895 kr.

b) $s_{12}=1000\cdot\frac{0,99^{12}-1}{0,99-1}\approx 11362$ Det første året får Heidi totalt 11362 kr.

c) $k\in\langle -1,1\rangle \Rightarrow$ konvergent rekke $s=\frac{1000}{1-0,99}=\frac{1000}{0,01}=100\,000$ Heidi får totalt 100 000 kr.

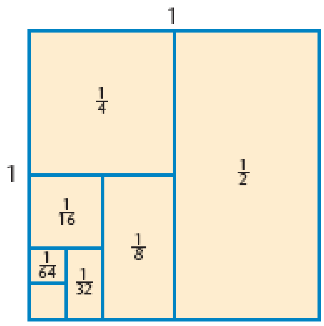
d) Hvis månedsbeløpet hadde økt med 1%, ville rekka blitt divergent da $k=1,01$.
Da ville Heidi fått 'uendelig' mye penger totalt.

Oppgave 6.73

a) Gitt den uendelige rekka: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$ Geometrisk rekke med kvotient $k=\frac{1}{2}$.

$k \in \langle -1,1\rangle \Rightarrow$ <u>Rekka konvergerer.</u> Summen blir: $s=\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{1}{2}}=\underline{\underline{1}}$

b)



Hele kvadratet har sidekanter med lengden 1.
Arealet av hele kvadratet er dermed $1 \cdot 1 = 1$

Vi deler så kvadratet som forklart i oppgaven. Arealet av rutene med tall på er til sammen

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$$

Denne summen er arealet av hele kvadratet bortsett fra ruta nederst til venstre. Summen er dermed

1 − arealet av den lille ruta nederst til venstre

Nå fortsetter vi oppdelingen av kvadratet. Ved å dele lenge nok, kan vi få arealet av ruta nederst til venstre så nær null vi bare vil bare ved å dele mange nok ganger. Vi kan dermed få summen

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots$$

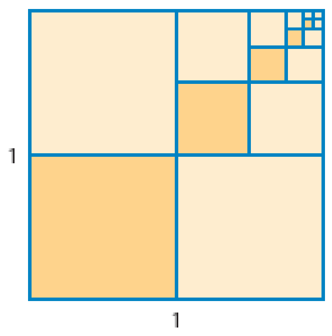
så nær 1 vi vil bare ved å ta med nok ledd.
Summen av den uendelige rekken er dermed 1.

Oppgave 6.74

a) Gitt den uendelige rekka: $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\ldots$ Geometrisk rekke med kvotient $k=\frac{1}{4}$.

$k \in \langle -1,1\rangle \Rightarrow$ <u>Rekka konvergerer.</u> Summen blir: $s=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{3}{4}}=\underline{\underline{\frac{1}{3}}}$

b)



De mørkeste rutene har samlet areal

$$\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{8}\cdot\frac{1}{8}+\ldots=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots$$

La S være summen av denne rekken.

Men de rutene som ligger under de mørke rutene har samlet areal

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots$$

Samlet areal av disse er dermed også S.

De rutene som ligger over de mørke rutene har også samlet areal

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots$$

Denne summen blir også S. Alle rutene til sammen utgjør hele kvadratet, som har arealet 1. Dermed må

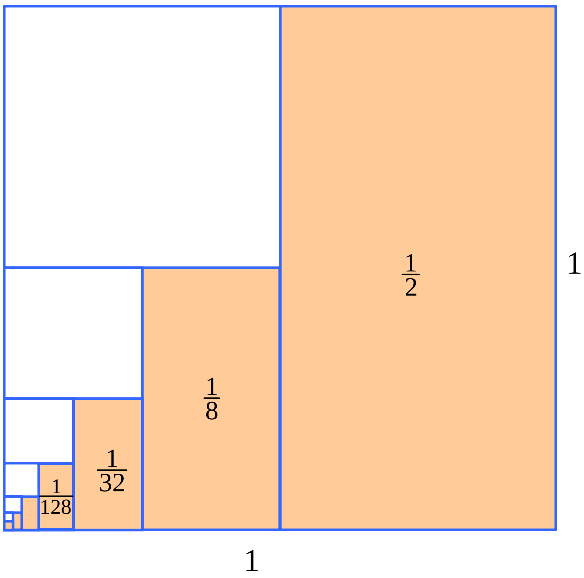
$$S+S+S=1$$
$$3S=1$$
$$S=\underline{\underline{\frac{1}{3}}}$$

Oppgave 6.75

a) Gitt den uendelige rekka: $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots$ Geometrisk rekke med kvotient $k=\frac{-\frac{1}{2}}{1}=-\frac{1}{2}$.

$k\in\langle -1,1\rangle \quad\Rightarrow\quad$ $\underline{\underline{\text{Rekka konvergerer.}}}$ Summen blir: $s=\frac{1}{1-\left(-\frac{1}{2}\right)}=\frac{1}{\frac{3}{2}}=\underline{\underline{\frac{2}{3}}}$

b)



$$1-\underbrace{\frac{1}{2}}_{=\frac{1}{2}}+\underbrace{\frac{1}{4}-\frac{1}{8}}_{=\frac{1}{8}}+\underbrace{\frac{1}{16}-\frac{1}{32}}_{=\frac{1}{32}}+\underbrace{\frac{1}{64}-\frac{1}{128}}_{=\frac{1}{128}}+\underbrace{\frac{1}{256}-\frac{1}{512}}_{=\frac{1}{512}}+\ldots$$

De lyse rutene har arealet

$$\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{8}\cdot\frac{1}{8}+\ldots=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots$$

Fra oppgave 6.74 vet vi at denne summen blir $\frac{1}{3}$.

Arealet av de mørke rutene blir da $1-\frac{1}{3}=\frac{2}{3}$.

Summen av den uendelige rekka

$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots$ blir dermed $\frac{2}{3}$

6.8 Geometriske rekker med variable kvotienter

Oppgave 6.80

a)

Gitt den uendelige rekken $x + x^3 + x^5 + x^7 + \dots \quad \Rightarrow \; k = \frac{x^3}{x} = x^2$

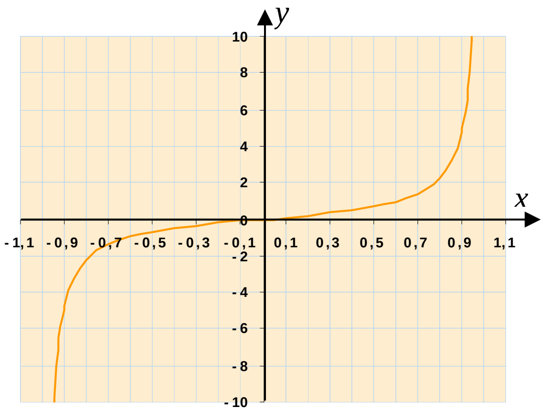
Rekken konvergerer når $-1 < x^2 < 1 \quad \Leftrightarrow \quad \underbrace{x^2 > -1}_{\text{Oppfylt for alle verdier av } x} \quad \wedge \quad \underbrace{x^2 < 1}_{\text{Oppfylt når } x \in \langle -1,1 \rangle}$

Rekken konvergerer når $x \in \langle -1, 1 \rangle$.

b)

Summen av rekken blir: $s(x) = \frac{x}{1-x^2}$

c)



Oppgave 6.81

a)

Gitt den uendelige rekken $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots \quad \Rightarrow \; k = \frac{\frac{1}{x}}{1} = \frac{1}{x}$

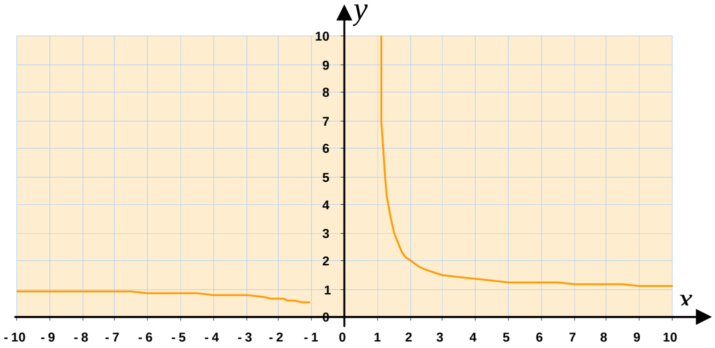
Rekken konvergerer når $-1 < \frac{1}{x} < 1 \quad \Leftrightarrow \quad \frac{1}{x} > -1 \quad \wedge \quad \frac{1}{x} < 1 \quad \Leftrightarrow \quad x < -1 \quad \wedge \quad x > 1$

Rekken konvergerer når $x \in \langle \leftarrow, -1 \rangle \cup \langle 1, \rightarrow \rangle$.

b)

Summen av rekken blir: $s(x) = \frac{1}{1-\frac{1}{x}} = \frac{x}{x-1}$

c)



d1) $s(x)=2 \quad \Rightarrow \quad \dfrac{x}{x-1}=2 \quad \Leftrightarrow \quad x=2\cdot(x-1) \quad \Leftrightarrow \quad x=2x-2 \quad \Leftrightarrow \quad 2x-x=2 \quad \Leftrightarrow \quad x=2$

Summen blir lik 2 når $x=2$.

d2) $s(x)=\frac{1}{3} \quad \Rightarrow \quad \dfrac{x}{x-1}=\frac{1}{3} \quad \Leftrightarrow \quad x=\frac{1}{3}\cdot(x-1) \quad \Leftrightarrow \quad 3x=x-1 \quad \Leftrightarrow \quad 2x=-1 \quad \Leftrightarrow$

$\underbrace{x=-\frac{1}{2}}_{\text{Ligger ikke i konvergensområdet}}$

Summen kan ikke bli $\frac{1}{3}$.

Oppgave 6.82

a) Gitt den uendelige rekken $1-\sin x+\sin^2 x+\sin^3 x+\ldots \quad , \quad x\in[0,2\pi]$

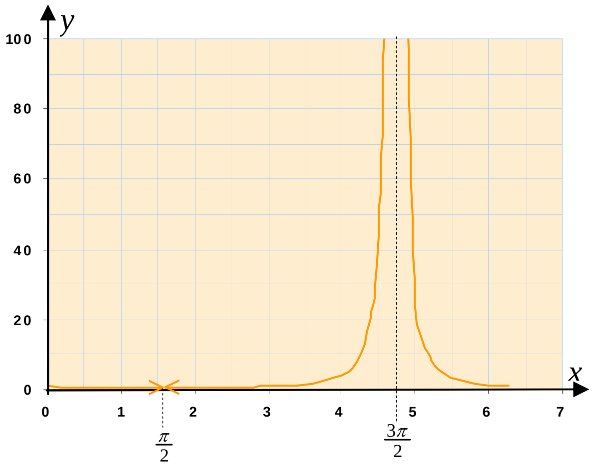
$\Rightarrow \quad k=\dfrac{-\sin x}{1}=-\sin x$

Rekken konvergerer når $-1<-\sin x<1 \quad \Leftrightarrow \quad \underbrace{\sin x<1}_{\substack{\text{Gjelder for alle } x \text{ i området} \\ \text{unntatt for } x=\frac{\pi}{2} \ (\sin\frac{\pi}{2}=1)}} \quad \wedge \quad \underbrace{\sin x>-1}_{\substack{\text{Gjelder for alle } x \text{ i området} \\ \text{unntatt for } x=\frac{3\pi}{2} \ (\sin\frac{3\pi}{2}=-1)}}$

Rekken konvergerer når $x\in[0,2\pi]\setminus\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$.

b) Summen av rekken blir: $s(x)=\dfrac{1}{1-(-\sin x)} \quad =\dfrac{1}{1+\sin x}$

c)



Oppgave 6.83

a) Gitt den uendelige rekken $1+2x+4x^2+8x^3+\ldots \quad \Rightarrow k=\frac{2x}{1}=2x$

Rekken konvergerer når $-1<2x<1 \quad \Leftrightarrow \quad -\frac{1}{2}<x<\frac{1}{2}$

Rekken konvergerer når $x\in\left\langle -\frac{1}{2},\frac{1}{2}\right\rangle$.

b) Summen av rekken blir: $s(x)=\frac{1}{1-2x}$

c1) $s(x)=2 \quad \Rightarrow \quad \frac{1}{1-2x}=2 \quad \Leftrightarrow \quad 1=2\cdot(1-2x) \quad \Leftrightarrow \quad 1=2-4x \quad \Leftrightarrow \quad 4x=1 \quad \Leftrightarrow \quad x=\frac{1}{4}$

Summen blir lik 2 når $x=\frac{1}{4}$.

c2) $s(x)=\frac{1}{3} \quad \Rightarrow \quad \frac{1}{1-2x}=\frac{1}{3} \quad \Leftrightarrow \quad 1=\frac{1}{3}\cdot(1-2x) \quad \Leftrightarrow \quad 3=1-2x \quad \Leftrightarrow \quad 2x=-2 \quad \Leftrightarrow$

$\underbrace{x=-1}_{\text{Utenfor konvergensområdet}}$

Summen kan ikke bli $\frac{1}{3}$.

6.9 Induksjonsbevis

Oppgave 6.90

Skal bruke induksjon til å vise at $1+2+3+...+n=\frac{n\cdot(n+1)}{2}$

Trinn 1: Viser at formelen er rett for $n=1$.

Venstre side = 1

Høyre side = $\frac{1\cdot(1+1)}{2}=\frac{1\cdot 2}{2}=1$

V.S=H.S og formelen er rett for $n=1$.

Trinn 2: Antar at formelen er rett for $n=k$, altså at $1+2+3+...+k=\frac{k\cdot(k+1)}{2}$

Må så vise at formelen også er rett for $n=k+1$,

altså at $1+2+3+...+(k+1)=\frac{(k+1)\cdot((k+1)+1)}{2}=\frac{(k+1)\cdot(k+2)}{2}$

$$1+2+3+...+k+(k+1)=\frac{k\cdot(k+1)}{2}+(k+1) \Leftrightarrow 1+2+3+...+k+(k+1)=\frac{\overbrace{k\cdot(k+1)+(k+1)\cdot 2}^{(k+1)\text{ er felles faktor}}}{2} \Leftrightarrow$$

$$1+2+3+...+(k+1)=\frac{(k+1)\cdot(k+2)}{2}$$

Formelen er derfor riktig også for $n=k+1$.

Formelen $1+2+3+...+n=\frac{n\cdot(n+1)}{2}$ er derfor riktig for alle heltallige $n\geq 1$.

Oppgave 6.91

Skal bruke induksjon til å vise at $1+2+2^2+\ldots+2^{n-1}=2^n-1$

Trinn 1: Viser at formelen er rett for $n=1$.

Venstre side = 1
Høyre side = $2^1-1=1$
} V.S=H.S og formelen er rett for $n=1$.

Trinn 2: Antar at formelen er rett for $n=k$, altså at $1+2+2^2+\ldots+2^{k-1}=2^k-1$
Må så vise at formelen også er rett for $n=k+1$,
altså at $1+2+2^2+\ldots+2^k=2^{k+1}-1$

$$1+2+2^2+\ldots+2^{k-1}+2^k=2^k-1+2^k \quad \Leftrightarrow \quad 1+2+2^2+\ldots+2^{k-1}+2^k=2\cdot 2^k-1 \quad \Leftrightarrow$$

$$1+2+2^2+\ldots+2^{k-1}+2^k=2^1\cdot 2^k-1 \quad \Leftrightarrow \quad 1+2+2^2+\ldots+2^k=2^{k+1}-1$$

Formelen er derfor riktig også for $n=k+1$.

Formelen $1+2+2^2+\ldots+2^{n-1}=2^n-1$ er derfor riktig for alle heltallige $n\geq 1$.

Oppgave 6.92

Skal bruke induksjon til å vise at $\left(x^n\right)' = n \cdot x^{n-1}$

Trinn 1: Viser at formelen er rett for $n = 1$.

Venstre side: $\left(x^1\right)' = 1$

Høyre side: $1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$

V.S=H.S og formelen er rett for $n = 1$.

Trinn 2: Antar at formelen er rett for $n = k$, altså at $\left(x^k\right)' = n \cdot x^{k-1}$

Må så vise at formelen også er rett for $n = k + 1$,

altså at $\left(x^{k+1}\right)' = (k+1) \cdot x^k$

$$\left(x^{k+1}\right)' = \left(x^k \cdot x\right)' = \left(x^k\right)' \cdot x + x^k \cdot x' \;\Leftrightarrow\; \left(x^{k+1}\right)' = k \cdot x^{k-1} \cdot x + x^k \cdot 1 \;\Leftrightarrow\; \left(x^k \cdot x\right)' = k \cdot x^{k-1} \cdot x^1 + x^k \;\Leftrightarrow$$

$$\left(x^{k+1}\right)' = k \cdot x^{k-1+1} + x^k \;\Leftrightarrow\; \left(x^{k+1}\right)' = k \cdot x^k + x^k \;\Leftrightarrow\; \left(x^{k+1}\right)' = x^k \cdot (k+1) \;\Leftrightarrow\; \left(x^{k+1}\right)' = (k+1) \cdot$$

Formelen er derfor riktig også for $n = k + 1$.

Formelen $\left(x^n\right)' = n \cdot x^{n-1}$ er derfor riktig for alle heltallige $n \geq 1$.

Oppgave 6.93

Skal bruke induksjon til å vise at $1^2+2^2+3^2+...+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Trinn 1: Viser at formelen er rett for $n=1$.

Venstre side = $1^2=1$

Høyre side = $\dfrac{1\cdot(1+1)(2\cdot1+1)}{6}=\dfrac{1\cdot2\cdot3}{6}=1$

V.S=H.S og formelen er rett for $n=1$.

Trinn 2: Antar at formelen er rett for $n=k$, altså at $1^2+2^2+3^2+...+k^2=\dfrac{k(k+1)(2k+1)}{6}$

Må så vise at formelen også er rett for $n=k+1$,

altså at $1^2+2^2+3^2+...+(k+1)^2=\dfrac{(k+1)((k+1)+1)(2\cdot(k+1)+1)}{6}=\dfrac{(k+1)(k+2)(2k+3)}{6}$

$$1^2+2^2+3^2+...+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2 \Leftrightarrow$$

$$1^2+2^2+3^2+...+(k+1)^2=\frac{\overbrace{k(k+1)(2k+1)+(k+1)^2\cdot6}^{(k+1)\text{ felles faktor}}}{6} \Leftrightarrow$$

$$1^2+2^2+3^2+...+(k+1)^2=\frac{(k+1)\cdot\left[k(2k+1)+(k+1)\cdot6\right]}{6} \Leftrightarrow$$

$$1^2+2^2+3^2+...+(k+1)^2=\frac{(k+1)\cdot\left[2k^2+k+6k+6\right]}{6} \Leftrightarrow$$

$$1^2+2^2+3^2+...+(k+1)^2=\frac{(k+1)\cdot\overbrace{\left[2k^2+7k+6\right]}^{=(k+2)(2k+3)}}{6}$$

$$1^2+2^2+3^2+...+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$$

Formelen er derfor riktig også for $n=k+1$.

Formelen $1^2+2^2+3^2+...+n^2=\dfrac{n(n+1)(2n+1)}{6}$ er derfor riktig for alle heltallige $n\geq1$.

Oppgave 6.94

Skal bruke induksjon til å vise at $s_n = n \cdot a_1 + \frac{n(n-1)}{2} \cdot d$ der s_n er summen av de n første leddene i en aritmetisk rekke med første ledd a_1 og differanse d.

Trinn 1: Viser at formelen er rett for $n = 1$.

Venstre side: a_1

Høyre side: $1 \cdot a_1 + \frac{1 \cdot (1-1)}{2} \cdot d = 1 \cdot a_1 + \frac{1 \cdot 0}{2} \cdot d = a_1$

V.S=H.S og formelen er rett for $n = 1$.

Trinn 2: Antar at formelen er rett for $n = k$, altså at $s_k = k \cdot a_1 + \frac{k(k-1)}{2} \cdot d$

Må så vise at formelen også er rett for $n = k+1$,

altså at $s_{k+1} = (k+1) \cdot a_1 + \frac{(k+1)k}{2} \cdot d$

$$s_k + a_{k+1} = k \cdot a_1 + \frac{k(k-1)}{2} \cdot d + a_{k+1} \quad \Leftrightarrow \quad s_k + a_{k+1} = k \cdot a_1 + \frac{k(k-1)}{2} \cdot d + (a_1 + k \cdot d) \quad \Leftrightarrow$$

$$s_k + a_{k+1} = k \cdot a_1 + a_1 + \frac{k(k-1)}{2} \cdot d + k \cdot d \quad \Leftrightarrow \quad s_k + a_{k+1} = (k+1) \cdot a_1 + \left(\frac{k(k-1)}{2} + k \right) \cdot d \quad \Leftrightarrow$$

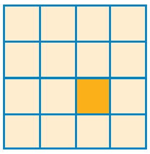
$$s_{k+1} = (k+1) \cdot a_1 + \frac{k(k-1) + k \cdot 2}{2} \cdot d \quad \Leftrightarrow \quad s_{k+1} = (k+1) \cdot a_1 + \frac{k(k-1) + 2k}{2} \cdot d \quad \Leftrightarrow$$

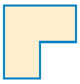
$$s_{k+1} = (k+1) \cdot a_1 + \frac{k(k-1+2)}{2} \cdot d \quad \Leftrightarrow \quad s_{k+1} = (k+1) \cdot a_1 + \frac{(k+1)k}{2} \cdot d$$

Formelen er derfor riktig også for $n = k+1$.

Formelen $s_n = n \cdot a_1 + \frac{n(n-1)}{2} \cdot d$ er derfor riktig for alle heltallige $n \geq 1$.

Oppgave 6.95





Oppgaven innebærer at antall hvite ruter på brettet dividert med 3 blir et helt tall.

Dette betyr at vi skal bruke induksjon til å vise at $\frac{\left(2^n\right)^2-1}{3}=\frac{2^{2n}-1}{3}=a\ ,\quad a\in N$

Trinn 1: Viser at formelen er rett for $n=1$.

$\frac{2^{2\cdot 1}-1}{3}=\frac{4-1}{3}=1\in N$ Formelen er rett for $n=1$.

Trinn 2: Antar at formelen er rett for $n=k$, altså at $\frac{2^{2k}-1}{3}=a\ ,\quad a\in N$

Må så vise at formelen også er rett for $n=k+1$,

altså at $\frac{2^{2(k+1)}-1}{3}=b\ ,\quad b\in N$

$$\frac{2^{2(k+1)}-1}{3}=\frac{2^{2k+2}-1}{3}=\frac{2^{2k}\cdot 2^2-1}{3}=\frac{4\cdot 2^{2k}-1}{3}=\frac{4\cdot\left(2^{2k}-1\right)+3}{3}=4\cdot\frac{2^{2k}-1}{3}+1=\underbrace{4a+1}_{b}$$

$a\in N \quad\Leftrightarrow\quad 4a+1\in N \quad\Rightarrow\quad \frac{2^{2(k+1)}-1}{3}=b\ ,\quad b\in N$

Formelen er derfor riktig også for $n=k+1$.

Formelen $\frac{\left(2^n\right)^2-1}{3}=\frac{2^{2n}-1}{3}=a\ ,\quad a\in N$ er derfor riktig for alle heltallige $n\geq 1$.

7.1 Noen integrasjonsformler

Oppgave 7.10

a) $\int 5\cos x dx = \underline{\underline{5\sin x + C}}$

b) $\int \left(\sin x + \sqrt{3}\cos x\right) dx = \underline{\underline{-\cos x + \sqrt{3}\sin x + C}}$

c) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x) dx = \left[-\cos x + \sin x\right]_0^{\frac{\pi}{2}} = \left(-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \left(-\cos 0 + \sin 0\right) = 0 + 1 + 1 - 0 = \underline{\underline{2}}$

d) $\int_0^{\frac{\pi}{4}} \frac{3}{\cos^2 t} dt = \left[3\tan t\right]_0^{\frac{\pi}{4}} = 3\tan\frac{\pi}{4} - 3\tan 0 = 3 \cdot 1 - 3 \cdot 0 = \underline{\underline{3}}$

Oppgave 7.11

a) $\int 4e^{2x+1} dx = 4 \cdot \frac{1}{2} \cdot e^{2x+1} + C = \underline{\underline{2e^{2x+1} + C}}$

b) $\int \frac{1}{x-2} dx = \frac{1}{1} \cdot \ln|x-2| + C = \underline{\underline{\ln|x-2| + C}}$

c) $\int \frac{1}{3x+1} dx = \underline{\underline{\frac{1}{3} \cdot \ln|3x+1| + C}}$

d) $\int \frac{3}{2-x} dx = 3 \cdot \frac{1}{-1} \cdot \ln|2-x| + C = \underline{\underline{-3\ln|2-x| + C}}$

Oppgave 7.12

a) $\int_0^1 2\sin(2\pi x) dx = \left[2 \cdot \frac{1}{2\pi} \cdot \left(-\cos(2\pi x)\right)\right]_0^1 = \left[-\frac{1}{\pi} \cdot \cos(2\pi x)\right]_0^1 = -\frac{1}{\pi} \cdot \cos(2\pi \cdot 1) - \left(-\frac{1}{\pi} \cdot \cos(2\pi \cdot 0)\right)$

$= -\frac{1}{\pi} \cdot 1 + \frac{1}{\pi} \cdot 1 = \underline{\underline{0}}$

b) $\int_1^3 \frac{1}{x+2} dx = \left[\ln|x+2|\right]_1^3 = \ln|3+2| - \ln|1+2| = \underline{\underline{\ln 5 - \ln 3}}$

c) $\int_1^2 2\sin\left(\frac{\pi}{3}x - \frac{\pi}{6}\right) dx = \left[2 \cdot \frac{1}{\frac{\pi}{3}} \cdot \left(-\cos\left(\frac{\pi}{3}x - \frac{\pi}{6}\right)\right)\right]_1^2 = \left[-\frac{6}{\pi} \cdot \cos\left(\frac{\pi}{3}x - \frac{\pi}{6}\right)\right]_1^2$

$= -\frac{6}{\pi}\cos\left(\frac{2\pi}{3} - \frac{\pi}{6}\right) - \left(-\frac{6}{\pi}\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)$

$= -\frac{6}{\pi}\cos\frac{\pi}{2} + \frac{6}{\pi}\cos\frac{\pi}{6} = -\frac{6}{\pi} \cdot 0 + \frac{6}{\pi} \cdot \frac{\sqrt{3}}{2} = \underline{\underline{\frac{3\sqrt{3}}{\pi}}}$

d) $\int_0^1 \frac{4}{2x+1} dx = \left[4 \cdot \frac{1}{2} \cdot \ln|2x+1|\right]_0^1 = 2\ln|2 \cdot 1 + 1| - 2\ln|2 \cdot 0 + 1| = 2\ln 3 - 2\ln 1 = \underline{\underline{2\ln 3}}$

7.2 Integrasjon ved variabelskifte

Oppgave 7.20

a) $\int 4e^{2x+1}dx$ Innfører $u = 2x+1 \;\Rightarrow\; \dfrac{du}{dx} = 2 \;\;|\cdot 2dx \;\Leftrightarrow\; 2du = 4dx$

$$\int 4e^{2x+1}dx = \int e^{2x+1}\cdot 4dx = \int e^{u}\cdot 2du = 2\int e^{u}du = 2e^{u} + C = \underline{\underline{2e^{2x+1} + C}}$$

b) $\int 6\pi \sin(2\pi x)\,dx$ Innfører $u = 2\pi x \;\Rightarrow\; \dfrac{du}{dx} = 2\pi \;\;|\cdot 3dx \;\Leftrightarrow\; 3du = 6\pi dx$

$$\int 6\pi\sin(2\pi x)\,dx = \int \sin(2\pi x)\cdot 6\pi dx = \int \sin u\cdot 3du = 3\int \sin u\,du = 3\cdot(-\cos u) + C = \underline{\underline{-3\cos(2\pi x) + C}}$$

c) $\int \frac{1}{3x+1}dx$ Innfører $u = 3x+1 \;\Rightarrow\; \dfrac{du}{dx} = 3 \;\;|\cdot \frac{1}{3}dx \;\Leftrightarrow\; \frac{1}{3}du = dx$

$$\int \frac{1}{3x+1}dx = \int \frac{1}{u}\cdot\frac{1}{3}du = \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + C = \underline{\underline{\frac{1}{3}\ln|3x+1| + C}}$$

Oppgave 7.21

a) $\int \frac{8x}{2x^2+5}dx$ Innfører $u = 2x^2 + 5 \;\Rightarrow\; \dfrac{du}{dx} = 4x \;\;|\cdot 2dx \;\Leftrightarrow\; 2du = 8xdx$

$$\int \frac{8x}{2x^2+5}dx = \int \frac{1}{2x^2+5}\cdot 8xdx = \int \frac{1}{u}\cdot 2du = 2\int \frac{1}{u}du = 2\ln|u| + C = 2\ln\Big|\underbrace{2x^2+5}_{\text{Alltid}>0}\Big| + C = \underline{\underline{2\ln\left(2x^2+5\right) + C}}$$

b) $\int \frac{6x^2}{x^3+1}dx$ Innfører $u = x^3 + 1 \;\Rightarrow\; \dfrac{du}{dx} = 3x^2 \;\;|\cdot 2dx \;\Leftrightarrow\; 2du = 6x^2dx$

$$\int \frac{6x^2}{x^3+1}dx = \int \frac{1}{x^3+1}\cdot 6x^2dx = \int \frac{1}{u}\cdot 2du = 2\int \frac{1}{u}du = 2\ln|u| + C = \underline{\underline{2\ln\left|x^3+1\right| + C}}$$

c) $\int \frac{2x}{\left(x^2+3\right)^3}dx$ Innfører $u = x^2 + 3 \;\Rightarrow\; \dfrac{du}{dx} = 2x \;\;|\cdot dx \;\Leftrightarrow\; du = 2xdx$

$$\int \frac{2x}{\left(x^2+3\right)^3}dx = \int \frac{1}{\left(x^2+3\right)^3}\cdot 2xdx = \int \frac{1}{u^3}du = \int u^{-3}du = \frac{1}{-2}u^{-2} + C = -\frac{1}{2}\cdot\frac{1}{u^2} + C = \underline{\underline{-\frac{1}{2}\cdot\frac{1}{\left(x^2+3\right)^2} + C}}$$

d) $\int x\cdot\left(x^2+1\right)^2 dx$ Innfører $u = x^2 + 1 \;\Rightarrow\; \dfrac{du}{dx} = 2x \;\;|\cdot \frac{1}{2}dx \;\Leftrightarrow\; \frac{1}{2}du = xdx$

$$\int x\cdot\left(x^2+1\right)^2 dx = \int \left(x^2+1\right)^2\cdot xdx = \int u^2\cdot\frac{1}{2}du = \frac{1}{2}\int u^2du = \frac{1}{2}\cdot\frac{1}{3}u^3 + C = \underline{\underline{\frac{1}{6}\left(x^2+1\right)^3 + C}}$$

Oppgave 7.22

a) $\int \cos x \cdot e^{\sin x}\,dx$ Innfører $u = \sin x \;\Rightarrow\; \frac{du}{dx} = \cos x \;|\cdot dx \;\Leftrightarrow\; du = \cos x dx$

$$\int \cos x \cdot e^{\sin x} dx = \int e^{\sin x} \cdot \cos x dx = \int e^u du = e^u + C = \underline{\underline{e^{\sin x} + C}}$$

b) $\int 10x\left(x^2+1\right)^4 dx$ Innfører $u = x^2+1 \;\Rightarrow\; \frac{du}{dx} = 2x \;|\cdot 5dx \;\Leftrightarrow\; 5du = 10xdx$

$$\int 10x\left(x^2+1\right)^4 dx = \int \left(x^2+1\right)^4 \cdot 10xdx = \int u^4 \cdot 5du = \int 5u^4 du = u^5 + C = \underline{\underline{\left(x^2+1\right)^5 + C}}$$

c) $\int (4x+4)e^{x^2+2x+1} dx$ Innfører $u = x^2+2x+1 \;\Rightarrow\; \frac{du}{dx} = 2x+2 \;|\cdot 2dx \;\Leftrightarrow\; 2du = (4x+4)dx$

$$\int (4x+4)e^{x^2+2x+1} dx = \int e^{x^2+2x+1} \cdot (4x+4)dx = \int e^u \cdot 2du = 2e^u + C = \underline{\underline{2e^{x^2+2x+1} + C}}$$

d) $\int \frac{e^x}{e^x+1} dx$ Innfører $u = e^x+1 \;\Rightarrow\; \frac{du}{dx} = e^x \;|\cdot dx \;\Leftrightarrow\; du = e^x dx$

$$\int \frac{e^x}{e^x+1} dx = \int \frac{1}{e^x+1} \cdot e^x dx = \int \frac{1}{u} du = \ln|u| + C = \ln\Big|\underbrace{e^x+1}_{\text{Alltid } >0}\Big| + C = \underline{\underline{\ln\left(e^x+1\right) + C}}$$

Oppgave 7.23

a) $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$ Innfører $u = \cos x \;\Rightarrow\; \frac{du}{dx} = -\sin x \;|\cdot(-1)dx \;\Leftrightarrow\; (-1)du = \sin x dx$

$$\int \tan x dx = \int \frac{1}{\cos x} \cdot \sin x dx = \int \frac{1}{u} \cdot (-1) du = -\ln|u| + C = \underline{\underline{-\ln|\cos x| + C}}$$

b)

$$\int_0^{\frac{\pi}{3}} \tan x dx = \left[-\ln|\cos x|\right]_0^{\frac{\pi}{3}} = -\ln\left|\cos\tfrac{\pi}{3}\right| - \left(-\ln|\cos 0|\right) = -\ln\tfrac{1}{2} + \ln 1 = -(\ln 1 - \ln 2) + 0 = \underline{\underline{\ln 2}}$$

Oppgave 7.24

$\int \frac{1}{1+\sqrt{x}} dx$ Innfører $u = 1+\sqrt{x} \;\Leftrightarrow\; \sqrt{x} = u-1$

$$\Rightarrow\; \frac{du}{dx} = \frac{1}{2\sqrt{x}} \;|\cdot 2\sqrt{x}dx \;\Leftrightarrow\; 2\sqrt{x}du = dx \;\Leftrightarrow\; dx = 2(u-1)du = (2u-2)du$$

$$\int \frac{1}{1+\sqrt{x}} dx = \int \frac{1}{u} \cdot (2u-2)du = \int \left(2 - \frac{2}{u}\right) du = 2u - 2\ln|u| + C' = 2\cdot\left(1+\sqrt{x}\right) - 2\ln\Big|\underbrace{1+\sqrt{x}}_{\text{Alltid } >0}\Big| + C'$$

$$= 2 + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C' \overset{2+C'=C}{=} \underline{\underline{2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C}}$$

7.3 Delvis integrasjon

Oppgave 7.30

a) $\int \overset{u}{x}\overset{v'}{\cos x}dx = \overset{u}{x}\cdot\overset{v}{\sin x} - \int \overset{u'}{1}\cdot\overset{v}{\sin x}dx = x\sin x - (-\cos x) + C = \underline{\underline{x\sin x + \cos x + C}}$

b) $\int \overset{u}{(2x+1)}\overset{v'}{e^x}dx = \overset{u}{(2x+1)}\cdot\overset{v}{e^x} - \int \overset{u'}{2}\cdot\overset{v}{e^x}dx = (2x+1)e^x - 2e^x + C = (2x+1-2)e^x = \underline{\underline{(2x-1)e^x + C}}$

c) $\int \overset{u}{2x}\,\overset{v'}{e^{x+1}}dx = \overset{u}{2x}\cdot\overset{v}{e^{x+1}} - \int \overset{u'}{2}\cdot\overset{v}{e^{x+1}}dx = 2xe^{x+1} - 2\cdot\frac{1}{1}\cdot e^{x+1} + C = \underline{\underline{2(x-1)e^{x+1} + C}}$

d) $\int \overset{u}{4x}\,\overset{v'}{e^{2x}}dx = \overset{u}{4x}\cdot\overset{v}{\tfrac{1}{2}e^{2x}} - \int \overset{u'}{4}\cdot\overset{v}{\tfrac{1}{2}e^{2x}}dx = 2xe^{2x} - 2\cdot\frac{1}{2}\cdot e^{2x} + C = \underline{\underline{(2x-1)e^{2x} + C}}$

Oppgave 7.31

a) $\int \overset{v'}{x}\,\overset{u}{\ln x}dx = \overset{v}{\tfrac{1}{2}x^2}\cdot\overset{u}{\ln x} - \int \overset{v}{\tfrac{1}{2}x^2}\cdot\overset{u'}{\tfrac{1}{x}}dx = \frac{1}{2}x^2\cdot\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\cdot\ln x - \frac{1}{2}\cdot\frac{1}{2}x^2 + C = \underline{\underline{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}}$

b) $\int \overset{v'}{(2x-1)}\,\overset{u}{\ln x}dx = \overset{v}{(x^2-x)}\cdot\overset{u}{\ln x} - \int \overset{v}{(x^2-x)}\cdot\overset{u'}{\tfrac{1}{x}}dx = (x^2-x)\cdot\ln x - \int (x-1)dx$

$= (x^2-x)\cdot\ln x - \left(\frac{1}{2}x^2 - x\right) + C = \underline{\underline{(x^2-x)\cdot\ln x - \frac{1}{2}x^2 + x + C}}$

c) $\int \overset{v'}{(6x^2+2x)}\,\overset{u}{\ln x}dx = \overset{v}{(2x^3+x^2)}\cdot\overset{u}{\ln x} - \int \overset{v}{(2x^3+x^2)}\cdot\overset{u'}{\tfrac{1}{x}}dx = (2x^3+x^2)\cdot\ln x - \int (2x^2+x)dx$

$= (2x^3+x^2)\cdot\ln x - \left(\frac{2}{3}x^3 + \frac{1}{2}x^2\right) + C = \underline{\underline{(2x^3+x^2)\cdot\ln x - \frac{2}{3}x^3 - \frac{1}{2}x^2 + C}}$

Oppgave 7.32

a) $\int \overset{u}{x^2}\,\overset{v'}{e^x}dx = \overset{u}{x^2}\cdot\overset{v}{e^x} - \int \overset{u'}{2x}\cdot\overset{v}{e^x}dx = x^2e^x - \int \overset{u}{2x}\cdot\overset{v'}{e^x}\,dx = x^2e^x - \left(\overset{u}{2x}\cdot\overset{v}{e^x} - \int \overset{u'}{2}\cdot\overset{v}{e^x}\,dx\right)$

$= x^2e^x - 2xe^x + 2e^x + C = \underline{\underline{(x^2 - 2x + 2)e^x + C}}$

b) $\int \overset{u}{x^2}\,\overset{v'}{\sin x}dx = \overset{u}{x^2}\cdot\overset{v}{(-\cos x)} + \int \overset{u'}{2x}\cdot\overset{v}{\cos x}dx = -x^2\cos x + \int \overset{u}{2x}\cdot\overset{v'}{\cos x}\,dx$

$= -x^2\cos x - \left(\overset{u}{2x}\cdot\overset{v}{(-\sin x)} - \int \overset{u'}{2}\cdot\overset{v}{(-\sin x)}\,dx\right)$

$= -x^2\cos x + 2x\sin x + 2\cos x + C = \underline{\underline{(2 - x^2)\cos x + 2x\sin x + C}}$

c)

$$\int \overset{u}{(2x^2+x-1)}\overset{v'}{\cos x}dx = \overset{u}{(2x^2+x-1)}\cdot\overset{v}{\sin x} - \int \overset{u'}{(4x+1)}\cdot\overset{v}{\sin x}dx = (2x^2+x-1)\sin x - \int \overset{u}{(4x+1)}\cdot\overset{v'}{\sin x}\,dx$$

$$= (2x^2+x-1)\sin x - \left(\overset{u}{(4x+1)}\cdot\overset{v}{(-\cos x)} - \int \overset{u'}{4}\cdot\overset{v}{(-\cos x)}\,dx\right)$$

$$= (2x^2+x-1)\sin x + (4x+1)\cos x + 4(-\sin x) + C$$

$$= (2x^2+x-1-4)\sin x + (4x+1)\cos x + C = \underline{\underline{(2x^2+x-5)\sin x + (4x+1)\cos x + C}}$$

Oppgave 7.33

a)

$$\int \overset{u}{(x^2+2x+1)}\overset{v'}{e^x}dx = \overset{u}{(x^2+2x+1)}\cdot\overset{v}{e^x} - \int \quad dx = (x^2+2x+1)e^x - \int \overset{u}{(2x+2)}\cdot\overset{v'}{e^x}dx$$

$$= (x^2+2x+1)e^x - \left[\overset{u}{(2x+2)}\cdot\overset{v}{e^x} - \int \overset{u'}{2}\cdot\overset{v}{e^x}\,dx\right]$$

$$= (x^2+2x+1)e^x - (2x+2)e^x + 2e^x + C =$$

$$= (x^2+2x+1-2x-2+2)e^x + C = \underline{\underline{(x^2+1)e^x + C}}$$

b)

$$\int_0^1 (x^2+2x+1)e^x dx = \left[(x^2+1)e^x\right]_0^1 = (1^2+1)e^1 - (0^2+1)e^0 = \underline{\underline{2e-1}}$$

Oppgave 7.34

$$\int \overset{u}{\cos x}\cdot\overset{v'}{e^x}dx = \overset{u}{\cos x}\cdot\overset{v}{e^x} - \int \overset{u'}{-\sin x}\cdot\overset{v}{e^x}dx = \cos x\cdot e^x + \int \overset{u}{\sin x}\cdot\overset{v'}{e^x}\,dx$$

$$= \cos x\cdot e^x + \left(\overset{u}{\sin x}\cdot\overset{v}{e^x} - \int \overset{u'}{\cos x}\cdot\overset{v}{e^x}\,dx\right) = \cos x\cdot e^x + \sin x\cdot e^x - \int \cos x\cdot e^x dx \quad \Leftrightarrow$$

$$\int \cos x\cdot e^x dx + \int \cos x\cdot e^x dx = \cos x\cdot e^x + \sin x\cdot e^x \quad \Leftrightarrow \quad 2\int \cos x\cdot e^x dx = \cos x\cdot e^x + \sin x\cdot e^x + C_1 \quad \Leftrightarrow$$

$$\int \cos x\cdot e^x dx = \frac{\cos x\cdot e^x + \sin x\cdot e^x + C_1}{2} = \underline{\underline{\tfrac{1}{2}(\cos x + \sin x)e^x + C}}$$

7.4 Bestemte integraler

Oppgave 7.40

a)

$$\int \overset{u}{x}\overset{v'}{\sin x}dx = \overset{u}{x}\cdot\overset{v}{(-\cos x)} - \int \overset{u'}{1}\cdot\overset{v}{(-\cos x)}dx = -x\cos x + \sin x + C$$

$$\int_0^{2\pi} x\sin x dx = \left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$= (-2\pi\cdot\cos 2\pi + \sin 2\pi) - (-0\cdot\cos 0 + \sin 0) = -2\pi\cdot 1 + 0 + 0 - 0 = \underline{\underline{-2\pi}}$$

Alternativt:

$$\int_0^{2\pi} \overset{u}{x}\overset{v'}{\sin x}dx = \left[\overset{u}{x}\cdot\overset{v}{(-\cos x)}\right]_0^{2\pi} - \int_0^{2\pi} \overset{u'}{1}\cdot\overset{v}{(-\cos x)}dx = (-2\pi\cdot\cos 2\pi - 0) + \left[\sin x\right]_0^{2\pi} = -2\pi + (\sin 2\pi - \sin 0)$$

$$= -2\pi + 0 - 0 = \underline{\underline{-2\pi}}$$

b)

$$\int \overset{u}{(x-1)}\overset{v'}{e^x}dx = \overset{u}{(x-1)}\cdot\overset{v}{e^x} - \int \overset{u'}{1}\cdot\overset{v}{e^x}dx = (x-1)e^x - e^x + C = (x-1-1)e^x + C = (x-2)e^x + C$$

$$\int_0^1 (x-1)e^x dx = \left[(x-2)e^x\right]_0^1 = (1-2)e^1 - (0-2)e^0 = -e - (-2)\cdot 1 = \underline{\underline{2-e}}$$

Alternativt:

$$\int_0^1 \overset{u}{(x-1)}\overset{v'}{e^x}dx = \left[\overset{u}{(x-1)}\cdot\overset{v}{e^x}\right]_0^1 - \int_0^1 \overset{u'}{1}\cdot\overset{v}{e^x}dx = \left((1-1)e^1 - (0-1)e^0\right) - \left[e^x\right]_0^1 = 1 - (e^1 - e^0) = 1 - e + 1 = \underline{\underline{2-e}}$$

Oppgave 7.41

a)

$$\int (2x+1)e^{x^2+x}dx \quad \text{Innfører } u = x^2 + x \;\Rightarrow\; \frac{du}{dx} = 2x+1 \;\Big|\cdot dx \;\Leftrightarrow\; du = (2x+1)dx$$

$$\int (2x+1)e^{x^2+x}dx = \int e^{x^2+x}\cdot(2x+1)dx = \int e^u du = e^u + C = e^{x^2+x} + C$$

$$\int_0^1 (2x+1)e^{x^2+x}dx = \left[e^{x^2+x}\right]_0^1 = e^{1^2+1} - e^{0^2-0} = \underline{\underline{e^2-1}}$$

Alternativt:

$$\left.\begin{array}{l} x = 1 \;\Rightarrow\; u = 1^2 + 1 = 2 \\ x = 0 \;\Rightarrow\; u = 0 \end{array}\right\} \quad \int_0^1 (2x+1)e^{x^2+x}dx = \int_0^2 e^u du = \left[e^u\right]_0^2 = e^2 - e^0 = \underline{\underline{e^2-1}}$$

b) $\int \frac{\cos x}{(\sin x+2)^2}dx$ Innfører $u = \sin x + 2 \Rightarrow \frac{du}{dx} = \cos x \,|\cdot dx \Leftrightarrow du = \cos x dx$

$$\int \frac{\cos x}{(\sin x+2)^2}dx = \int \frac{1}{(\sin x+2)^2}\cdot \cos x dx = \int \frac{1}{u^2}du = \int u^{-2}du = \frac{1}{-1}u^{-1} + C = -\frac{1}{u} + C = -\frac{1}{\sin x+2} + C$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(\sin x+2)^2}dx = \left[-\frac{1}{\sin x+2}\right]_0^{\frac{\pi}{2}} = -\frac{1}{\sin\frac{\pi}{2}+2} - \left(-\frac{1}{\sin 0+2}\right) = -\frac{1}{1+2} + \frac{1}{2} = -\frac{1}{3} + \frac{1}{2} = \frac{-2+3}{6} = \underline{\underline{\frac{1}{6}}}$$

Alternativt:

$$\left.\begin{array}{l} x = \frac{\pi}{2} \Rightarrow u = \sin\frac{\pi}{2} + 2 = 3 \\ x = 0 \Rightarrow u = \sin 0 + 2 = 2 \end{array}\right\} \int_0^{\frac{\pi}{2}} \frac{\cos x}{(\sin x+2)^2}dx = \int_2^3 u^{-2}du = \left[-\frac{1}{u}\right]_2^3 = -\frac{1}{3} - \left(-\frac{1}{2}\right) = -\frac{1}{3} + \frac{1}{2} = \underline{\underline{\frac{1}{6}}}$$

Oppgave 7.42

a) $\int 4x(x^2+1)e^{x^2+1}dx$ Innfører $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \,|\cdot 2dx \Leftrightarrow 2du = 4xdx$

$$\int 4x(x^2+1)e^{x^2+1}dx = \int (x^2+1)e^{x^2+1}\cdot 4xdx = \int ue^u\cdot 2du = \int \overset{v}{2u}\,\overset{w'}{e^u}du = \overset{v}{2u}\,\overset{w}{e^u} - \int \overset{v'}{2}\,\overset{w}{e^u}\,du$$

$$= 2ue^u - 2e^u + C = 2e^u(u-1) + C = 2e^{x^2+1}(x^2+1-1) + C = \underline{\underline{2x^2e^{x^2+1} + C}}$$

b) $\int_0^1 4x(x^2+1)e^{x^2+1}dx = \left[2x^2e^{x^2+1}\right]_0^1 = 2\cdot 1^2\cdot e^{1^2+1} - 2\cdot 0^2\cdot e^{0^2+1} = \underline{\underline{2e^2}}$

Oppgave 7.43

a) Samlet forbruk er gitt ved $\int_0^{12} S(x)dx$

$$\int_0^{12} S(x)dx = \int_0^{12}\left(2500 - 1500\sin\left(\frac{\pi}{6}x - \frac{2\pi}{3}\right)\right)dx = \left[2500x - 1500\cdot\frac{1}{\frac{\pi}{6}}\left(-\cos\left(\frac{\pi}{6}x - \frac{2\pi}{3}\right)\right)\right]_0^{12}$$

$$= \left[2500x + \frac{9000}{\pi}\cos\left(\frac{\pi}{6}x - \frac{2\pi}{3}\right)\right]_0^{12}$$

$$= \left(2500\cdot 12 + \frac{9000}{\pi}\cos\left(\frac{\pi}{6}\cdot 12 - \frac{2\pi}{3}\right)\right) - \left(2500\cdot 0 + \frac{9000}{\pi}\cos\left(\frac{\pi}{6}\cdot 0 - \frac{2\pi}{3}\right)\right)$$

$$= 30000 + \frac{9000}{\pi}\cdot\left(\cos\frac{4\pi}{3}\right) - 0 - \frac{9000}{\pi}\cos\left(-\frac{2\pi}{3}\right) = 30000 + \frac{9000}{\pi}\cdot\left(-\frac{1}{2}\right) - \frac{9000}{\pi}\cdot\left(-\frac{1}{2}\right)$$

$$= 30000 - \frac{9000}{2\pi} + \frac{9000}{2\pi} = 30000$$

<u>Samlet strømforbruk på ett år blir 30 000kWh.</u>

b) Gjennomsnittlig strømforbruk per måned: $\frac{30\,000\text{kWh}}{12} = \underline{\underline{2\,500\text{kWh}}}$

7.5 Delbrøkoppspalting

Oppgave 7.50

a) $\int \frac{x+4}{x^2+2x}\,dx = \int \frac{x+4}{x\cdot(x+2)}\,dx$

$\frac{x+4}{x\cdot(x+2)} = \frac{A}{x} + \frac{B}{x+2} \quad \Big| \cdot x\cdot(x+2) \quad \Leftrightarrow \quad x+4 = A\cdot(x+2) + B\cdot x$

$x=0 \quad \Rightarrow \quad 0+4 = A\cdot(0+2) + B\cdot 0 \quad \Leftrightarrow \quad 2A = 4 \quad \Leftrightarrow \quad A = 2$

$x=-2 \quad \Rightarrow \quad -2+4 = A\cdot(-2+2) + B\cdot(-2) \quad \Leftrightarrow \quad 2B = -2 \quad \Leftrightarrow \quad B = -1$

$\int \frac{x+4}{x^2+2x}\,dx = \int \left(\frac{2}{x} - \frac{1}{x+2}\right)dx = \underline{\underline{2\ln|x| - \ln|x+2| + C}}$

b)

$\int \frac{3x}{x^2-x-2}\,dx \overset{\substack{\text{Nullpunkter for nevneren:} \\ x_1=-1 \,\wedge\, x_2=2}}{=} \int \frac{3x}{(x+1)\cdot(x-2)}\,dx$

$\frac{3x}{(x+1)\cdot(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \quad \Big| \cdot(x+1)\cdot(x-2) \quad \Leftrightarrow \quad 3x = A\cdot(x-2) + B\cdot(x+1)$

$x=-1 \quad \Rightarrow \quad 3\cdot(-1) = A\cdot(-1-2) + B\cdot(-1+1) \quad \Leftrightarrow \quad 3A = 3 \quad \Leftrightarrow \quad A = 1$

$x=2 \quad \Rightarrow \quad 3\cdot 2 = A\cdot(2-2) + B\cdot(2+1) \quad \Leftrightarrow \quad 3B = 6 \quad \Leftrightarrow \quad B = 2$

$\int \frac{3x}{x^2-x-2}\,dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right)dx = \underline{\underline{\ln|x+1| + 2\ln|x-2| + C}}$

c)

$$\int \frac{6x^2-17x+6}{x^3-5x^2+6x}\,dx \overset{\substack{\text{Nullpunkter for}\\ \text{nevneren:}\\ x_1=0 \,\wedge\, x_2=2 \,\wedge\, x_3=3}}{=} \int \frac{6x^2-17x+6}{x\cdot(x-2)\cdot(x-3)}\,dx$$

$$\frac{6x^2-17x+6}{x\cdot(x-2)\cdot(x-3)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x-3} \quad \Big|\cdot x\cdot(x-2)\cdot(x-3) \quad \Leftrightarrow$$

$$6x^2-17x+6 = A\cdot(x-2)\cdot(x-3)+B\cdot x\cdot(x-3)+C\cdot x\cdot(x-2)$$

$$x=0 \;\Rightarrow\; 6\cdot 0^2-17\cdot 0+6 = A\cdot(0-2)\cdot(0-3)+B\cdot 0\cdot(0-3)+C\cdot 0\cdot(x-2) \;\Leftrightarrow\; 6A=6 \;\Leftrightarrow\; A=1$$

$$x=2 \;\Rightarrow\; 6\cdot 2^2-17\cdot 2+6 = A\cdot(2-2)\cdot(2-3)+B\cdot 2\cdot(2-3)+C\cdot 2\cdot(2-2) \;\Leftrightarrow\; 2B=4 \;\Leftrightarrow\; B=2$$

$$x=3 \;\Rightarrow\; 6\cdot 3^2-17\cdot 3+6 = A\cdot(3-2)\cdot(3-3)+B\cdot 3\cdot(3-3)+C\cdot 3\cdot(3-2) \;\Leftrightarrow\; 3C=9 \;\Leftrightarrow\; C=3$$

$$\int \frac{6x^2-17x+6}{x^3-5x^2+6x}\,dx = \int\left(\frac{1}{x}+\frac{2}{x-2}+\frac{3}{x-3}\right)dx = \underline{\underline{\ln|x|+2\ln|x-2|+3\ln|x-3|+C}}$$

Oppgave 7.51

a)

$$\int \frac{2x+4}{x^2+4x+3}\,dx \overset{\substack{\text{Nullpunkter for}\\ \text{nevneren:}\\ x_1=-1 \,\wedge\, x_2=-3}}{=} \int \frac{2x+4}{(x+1)\cdot(x+3)}\,dx$$

$$\frac{2x+4}{(x+1)\cdot(x+3)} = \frac{A}{x+1}+\frac{B}{x+3} \quad \Big|\cdot(x+1)\cdot(x+3) \quad \Leftrightarrow \quad 2x+4 = A\cdot(x+3)+B\cdot(x+1)$$

$$x=-1 \;\Rightarrow\; 2\cdot(-1)+4 = A\cdot(-1+3)+B\cdot(-1+1) \;\Leftrightarrow\; 2A=2 \;\Leftrightarrow\; A=1$$

$$x=-3 \;\Rightarrow\; 2\cdot(-3)+4 = A\cdot(-3+3)+B\cdot(-3+1) \;\Leftrightarrow\; 2B=2 \;\Leftrightarrow\; B=1$$

$$\int \frac{2x+4}{x^2+4x+3}\,dx = \int\left(\frac{1}{x+1}+\frac{1}{x+3}\right)dx = \underline{\underline{\ln|x+1|+\ln|x+3|+C}}$$

b)

$$\int \frac{3x-5}{x^2-x-12}\,dx \overset{\substack{\text{Nullpunkter for} \\ \text{nevneren:} \\ x_1=-3 \,\wedge\, x_2=4}}{=} \int \frac{3x-5}{(x+3)\cdot(x-4)}\,dx$$

$$\frac{3x-5}{(x+3)\cdot(x-4)} = \frac{A}{x+3} + \frac{B}{x-4} \quad \Big|\cdot(x+3)\cdot(x-4) \quad \Leftrightarrow \quad 3x-5 = A\cdot(x-4)+B\cdot(x+3)$$

$$x=-3 \;\Rightarrow\; 3\cdot(-3)-5 = A\cdot(-3-4)+B\cdot(-3+3) \;\Leftrightarrow\; 7A=14 \;\Leftrightarrow\; A=2$$

$$x=4 \;\Rightarrow\; 3\cdot 4-5 = A\cdot(4-4)+B\cdot(4+3) \;\Leftrightarrow\; 7B=7 \;\Leftrightarrow\; B=1$$

$$\int \frac{3x-5}{x^2-x-12}\,dx = \int\left(\frac{2}{x+3}+\frac{1}{x-4}\right)dx = \underline{\underline{2\ln|x+3| + \ln|x-4| + C}}$$

Oppgave 7.52

a) $\int \frac{2x^2+5x+1}{x^2+x}\,dx$

$\left(2x^2+5x+1\right):\left(x^2+x\right)=2+\frac{3x+1}{x^2+x}=2+\frac{3x+1}{x\cdot(x+1)}$

$\underline{2x^2+2x}$

$\qquad 3x+1$

$\frac{3x+1}{x\cdot(x+1)}=\frac{A}{x}+\frac{B}{x+1} \quad \Big|\cdot x\cdot(x+1) \quad \Leftrightarrow \quad 3x+1=A\cdot(x+1)+B\cdot x$

$x=-1 \;\Rightarrow\; 3\cdot(-1)+1=A\cdot(-1+1)+B\cdot(-1) \;\Leftrightarrow\; B=2$

$x=0 \;\Rightarrow\; 3\cdot 0+1=A\cdot(0+1)+B\cdot 0 \;\Leftrightarrow\; A=1$

$\int \frac{2x^2+5x+1}{x^2+x}\,dx=\int\left(2+\frac{1}{x}+\frac{2}{x+1}\right)dx=\underline{\underline{2x+\ln|x|+2\ln|x+1|+C}}$

b) $\int \frac{2x^3+x^2-2x-3}{x^2-1}\,dx$

$$\left(2x^3+x^2-2x-3\right):\left(x^2-1\right)=2x+1+\frac{-2}{x^2-1}=2x+1-\frac{2}{(x+1)(x-1)}$$

$$\underline{2x^3-2x}$$
$$x^2-3$$
$$\underline{x^2-1}$$
$$-2$$

$$\frac{2}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}\quad \Big|\cdot(x+1)\cdot(x-1)\quad \Leftrightarrow\quad 2=A\cdot(x-1)+B\cdot(x+1)$$

$$x=-1\ \Rightarrow\ 2=A\cdot(-1-1)+B\cdot(-1+1)\ \Leftrightarrow\ 2A=-2\ \Leftrightarrow\ A=-1$$
$$x=1\ \Rightarrow\ 2=A\cdot(1-1)+B\cdot(1+1)\ \Leftrightarrow\ 2B=2\ \Leftrightarrow\ B=1$$

$$\int \frac{2x^3+x^2-2x-3}{x^2-1}\,dx=\int\left(2x+1-\left(\frac{-1}{x+1}+\frac{1}{x-1}\right)\right)dx=\int\left(2x+1+\frac{1}{x+1}-\frac{1}{x-1}\right)dx$$
$$=x^2+x+\ln|x+1|-\ln|x-1|+C$$

$$\int_2^3 \frac{2x^3+x^2-2x-3}{x^2-1}\,dx=\left[x^2+x+\ln|x+1|-\ln|x-1|\right]_2^3$$
$$=\left(3^2+3+\ln|3+1|-\ln|3-1|\right)-\left(2^2+2+\ln|2+1|-\ln|2-1|\right)$$
$$=9+3+\underbrace{\ln 4}_{=2\ln 2}-\ln 2-4-2-\ln 3+\underbrace{\ln 1}_{=0}=\underline{\underline{6+\ln 2-\ln 3}}$$

7.6 Funksjonsdrøfting

Oppgave 7.60

a) Gitt funksjonen $f(x) = 4xe^{-x}$

Nullpunkter der $f(x) = 0 \quad \Rightarrow \quad 4xe^{-x} = 0 \quad \Leftrightarrow \quad 4x = 0 \quad \left(e^{-x} \neq 0\right) \quad \Leftrightarrow \quad x = 0$

Funksjonen har nullpunktet $x = 0$.

b) $f(x) = 4xe^{-x} \quad \Rightarrow \quad f'(x) = 4 \cdot e^{-x} + 4x \cdot e^{-x} \cdot (-1) = 4e^{-x} - 4xe^{-x} = 4e^{-x}(1-x)$

Toppunkt der $f'(x) = 0 \quad \Rightarrow \quad 4e^{-x}(1-x) = 0 \quad \Leftrightarrow \quad 1 - x = 0 \quad \left(4e^{-x} \neq 0\right) \quad \Leftrightarrow \quad x = 1$

$f(1) = 4 \cdot 1 \cdot e^{-1} = \frac{4}{e}$ Funksjonen har toppunkt i $\left(1, \frac{4}{e}\right)$.

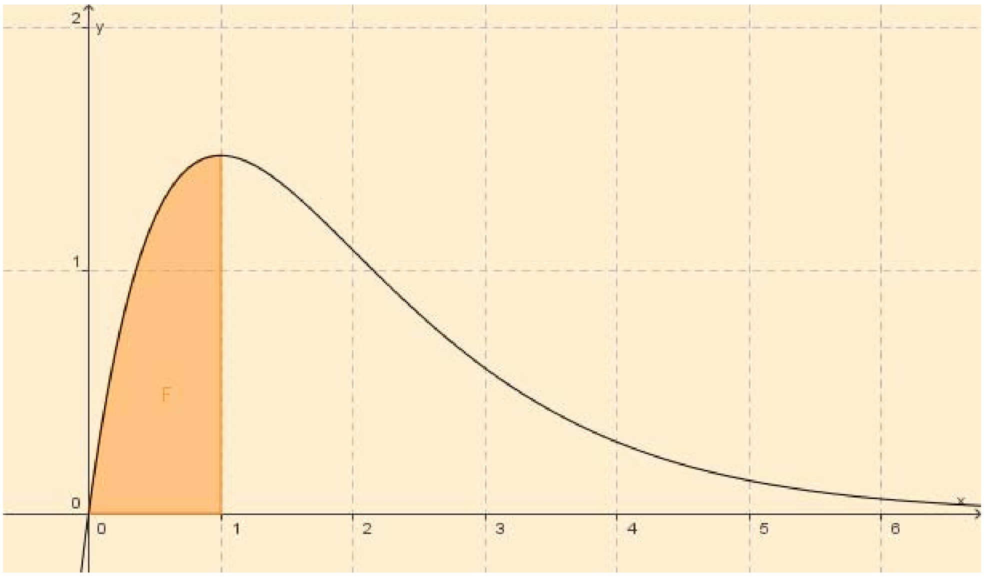
c) $f'(x) = 4e^{-x}(1-x) \quad \Rightarrow \quad f''(x) = 4e^{-x} \cdot (-1) \cdot (1-x) + 4e^{-x} \cdot (-1) = -4e^{-x}(1-x+1) = -4e^{-x}(2-x)$

Vendepunkt der $f''(x) = 0 \quad \Rightarrow$

$-4e^{-x}(2-x) = 0 \quad \Leftrightarrow \quad 2 - x = 0 \quad \left(-4e^{-x} \neq 0\right) \quad \Leftrightarrow \quad x = 2$

$f(2) = 4 \cdot 2 \cdot e^{-2} = \frac{8}{e^2}$ Funksjonen har vendepunkt i $\left(2, \frac{8}{e^2}\right)$.

d)



e) $F = \int_0^1 \overset{u}{4x}\,\overset{v'}{e^{-x}}\,dx = \left[\overset{u}{4x}\cdot\overset{v}{(-1)e^{-x}}\right]_0^1 - \int_0^1 \overset{u'}{4}\cdot\overset{v}{(-1)e^{-x}}\,dx = \left(-4\cdot 1\cdot e^{-1} - 0\right) - \left[4e^{-x}\right]_0^1 = -\frac{4}{e} - \left(\frac{4}{e} - 4\right) = 4 - \frac{8}{e}$

f)

$$V=\pi\int_0^1\left(4xe^{-x}\right)^2dx=\pi\int_0^1 16x^2e^{-2x}dx$$

$$\int \overset{u}{16x^2}\,\overset{v'}{e^{-2x}}dx=16x^2\cdot\left(-\tfrac{1}{2}\right)e^{-2x}-\int \overset{u'}{32x}\cdot\overset{v}{\left(-\tfrac{1}{2}\right)e^{-2x}}dx=-8x^2e^{-2x}+\int \overset{w}{16x}\,\overset{v'}{e^{-2x}}dx$$

$$=-8x^2e^{-2x}+\left(\overset{w}{16x}\cdot\left(-\tfrac{1}{2}\right)e^{-2x}-\int \overset{w'}{16}\cdot\overset{v}{\left(-\tfrac{1}{2}\right)e^{-2x}}dx\right)=-8x^2e^{-2x}-8xe^{-2x}+\int 8e^{-2x}dx$$

$$=-8x^2e^{-2x}-8xe^{-2x}-4e^{-2x}+C=-4e^{-2x}\cdot\left(2x^2+2x+1\right)+C$$

$$V=\pi\int_0^1 16x^2e^{-2x}dx=\pi\cdot\left[-4e^{-2x}\cdot\left(2x^2+2x+1\right)\right]_0^1$$

$$=\pi\left[\left(-4e^{-2\cdot1}\cdot\left(2\cdot1^2+2\cdot1+1\right)\right)-\left(-4e^{0}\cdot\left(2\cdot0^2+2\cdot0+1\right)\right)\right]$$

$$=\pi\left[-\frac{4}{e^2}\cdot5-(-4)\right]=\pi\left(-\frac{20}{e^2}+4\right)=\underline{\underline{4\pi-\frac{20\pi}{e^2}}}$$

Oppgave 7.61

a) Gitt funksjonen $f(x)=\dfrac{4x}{x^2+1}$

Nullpunkter der $f(x)=0 \quad\Rightarrow\quad \dfrac{4x}{x^2+1}=0 \quad\Leftrightarrow\quad 4x=0 \quad\Leftrightarrow\quad x=0$

Funksjonen har nullpunktet $\underline{\underline{x=0}}$.

b) Ingen vertikal asymptote da $x^2+1\neq0$.

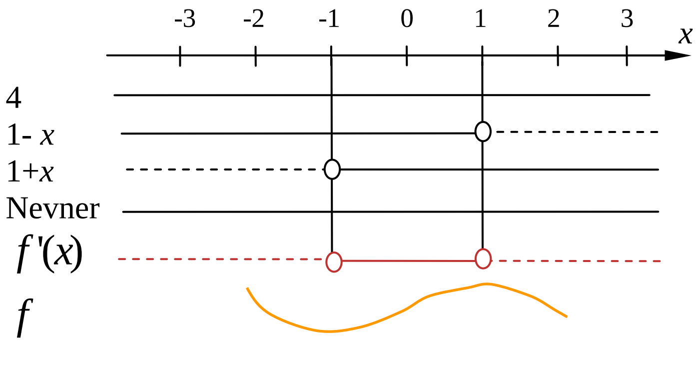
$$x\to\infty \quad\Rightarrow\quad f(x)=\frac{4x}{x^2+1}=\frac{\frac{4x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}=\frac{\frac{4}{x}}{1+\frac{1}{x^2}}\to\frac{0}{1+0}=0$$

Horisontal asymptote $\underline{\underline{y=0}}$.

c)

$$f(x)=\frac{4x}{x^2+1}\quad\Rightarrow$$

$$f'(x)=\frac{4\cdot\left(x^2+1\right)-4x\cdot2x}{\left(x^2+1\right)^2}=\frac{4x^2+4-8x^2}{\left(x^2+1\right)^2}=\frac{4-4x^2}{\left(x^2+1\right)^2}=\frac{4\left(1-x^2\right)}{\left(x^2+1\right)^2}=\frac{4\cdot(1-x)\cdot(1+x)}{\left(x^2+1\right)^2}$$



$$f(-1)=\frac{4\cdot(-1)}{(-1)^2+1}=\frac{-4}{2}=-2$$

$$f(1)=\frac{4\cdot 1}{1^2+1}=\frac{4}{2}=2$$

Bunnpunkt i $(-1,-2)$

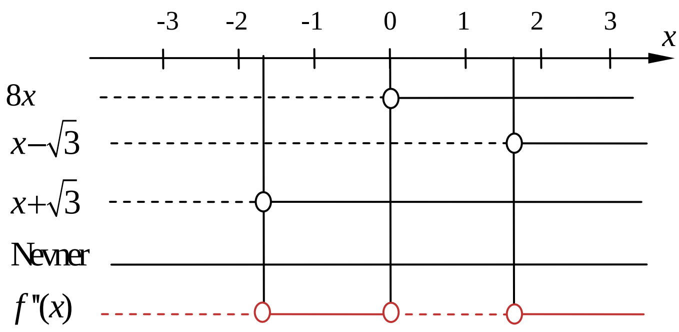
Toppunkt i $(1,2)$

d)

$$f'(x)=\frac{4-4x^2}{\left(x^2+1\right)^2}\quad\Rightarrow$$

$$f''(x)=\frac{-8x\cdot\left(x^2+1\right)^2-\left(4-4x^2\right)\cdot 2\left(x^2+1\right)\cdot 2x}{}=\frac{4x\left(x^2+1\right)\cdot\left[-2\left(x^2+1\right)-\left(4-4x^2\right)\right]}{\left(x^2+1\right)^4}$$

$$=\frac{4x\left(-2x^2-2-4+4x^2\right)}{\left(x^2+1\right)^3}=\frac{4x\cdot\left(2x^2-6\right)}{\left(x^2+1\right)^3}=\frac{8x\cdot\left(x^2-3\right)}{\left(x^2+1\right)^3}=\frac{8x\cdot\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)}{\left(x^2+1\right)^3}$$

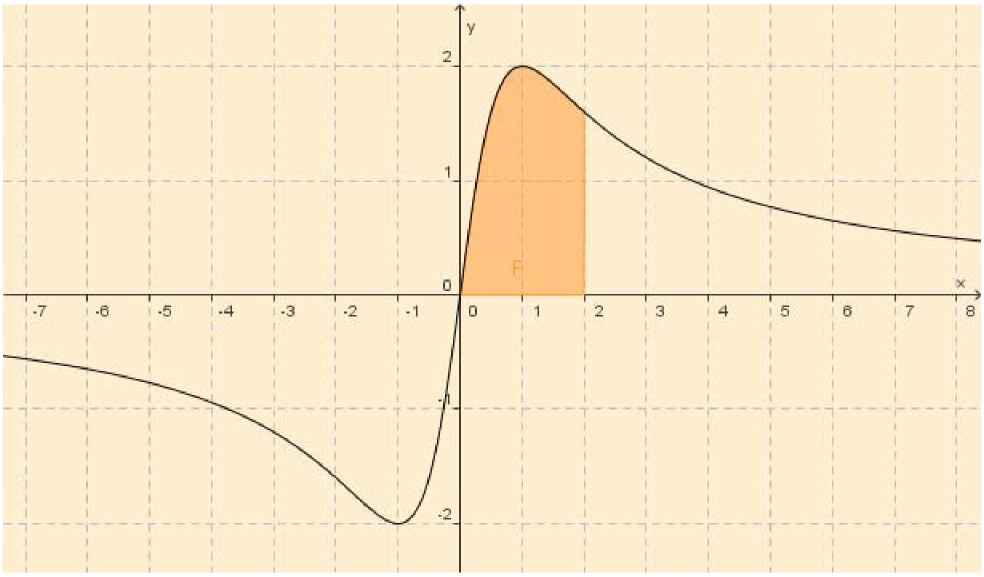


$$f\left(-\sqrt{3}\right)=\frac{4\cdot\left(-\sqrt{3}\right)}{\left(-\sqrt{3}\right)^2+1}=\frac{-4\sqrt{3}}{4}=-\sqrt{3}\qquad f(0)=\frac{4\cdot 0}{0^2+1}=\frac{0}{1}=0$$

$$f\left(\sqrt{3}\right)=\frac{4\cdot\sqrt{3}}{\left(\sqrt{3}\right)^2+1}=\frac{4\sqrt{3}}{4}=\sqrt{3}$$

Vendepunkter i $\left(-\sqrt{3},-\sqrt{3}\right)$, $(0,0)$ og $\left(\sqrt{3},\sqrt{3}\right)$.

e)



f)

$$F=\int_0^2 \frac{4x}{x^2+1}\,dx$$

Innfører $u=x^2+1 \quad \Rightarrow \quad \frac{du}{dx}=2x \;\; |\cdot 2dx \quad \Leftrightarrow \quad 2du=4xdx$

$x=0 \quad \Rightarrow \quad u=0^2+1=1$

$x=2 \quad \Rightarrow \quad u=2^2+1=5$

$$F=\int_0^2 \frac{4x}{x^2+1}\,dx=\int_0^2 \frac{1}{x^2+1}\cdot 4xdx=\int_1^5 \frac{1}{u}\cdot 2du=\left[2\ln|u|\right]_1^5=2\ln 5-2\ln 1=\underline{\underline{2\ln 5}}$$

Oppgave 7.62

a) Vertikal asymptote der $x^2+x-6=0 \;\Rightarrow$

$$x=\frac{-1\pm\sqrt{1^2-4\cdot 1\cdot(-6)}}{2\cdot 1}=\frac{-1\pm\sqrt{25}}{2} \quad \Leftrightarrow \quad x=-3 \;\vee\; x=2$$

$\underline{\underline{\text{Vertikale asymptoter der } x=-3 \text{ og } x=2.}}$

$$x\to\infty \Rightarrow f(x)=\frac{5x}{x^2+x-6}=\frac{\frac{5x}{x^2}}{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{6}{x^2}}=\frac{\frac{5}{x}}{1+\frac{1}{x}-\frac{6}{x^2}}\to\frac{0}{1+0-0}=0$$

$\underline{\underline{\text{Horisontal asymptote } y=0.}}$

b) $f(1)=\dfrac{5\cdot 1}{1^2+1-6}=\dfrac{5}{-4}=-\dfrac{5}{4}$ Tangeringspunkt $\left(1,-\dfrac{5}{4}\right)$.

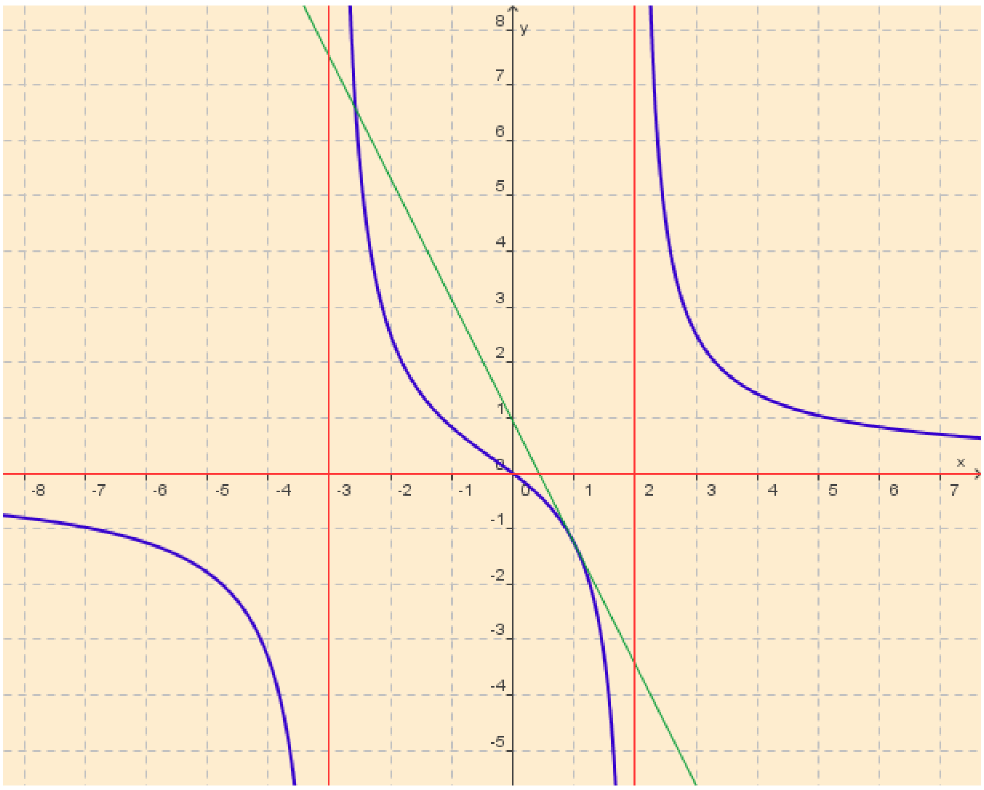
$$f(x)=\frac{5x}{x^2+x-6} \quad \Rightarrow$$

$$f'(x)=\frac{5\cdot\left(x^2+x-6\right)-5x\cdot(2x+1)}{\left(x^2+x-6\right)^2}=\frac{5x^2+5x-30-10x^2-5x}{\left(x^2+x-6\right)^2}=\frac{-5x^2-30}{\left(x^2+x-6\right)^2}$$

Tangentens stigningstall: $a=f'(1)=\dfrac{-5\cdot 1^2-30}{\left(1^2+1-6\right)^2}=\dfrac{-35}{16}=-\dfrac{35}{16}$

$y=-\dfrac{35}{16}x+b \quad \Rightarrow \quad -\dfrac{5}{4}=-\dfrac{35}{16}\cdot 1+b \quad \Leftrightarrow \quad b=\dfrac{15}{16}$ Tangenten gitt ved $y=-\dfrac{35}{16}x+\dfrac{15}{16}$.

c)



d) Skjæringspunkter gitt ved $\frac{5x}{x^2+x-6}=-\frac{35}{16}x+\frac{15}{16} \Leftrightarrow 5x\cdot 16=(-35x+15)\cdot(x^2+x-6)$

$\Leftrightarrow 80x=-35x^3-35x^2+210x+15x^2+15x-90 \Leftrightarrow 35x^3+20x^2-145x+90=0$

Vet at $x=1$ er en løsning på likningen, derfor kan vi utføre polynomdivisjonen

$(35x^3+20x^2-145x+90):(x-1)=35x^2+55x-90$

$$\begin{array}{l} \underline{35x^3-35x^2} \\ \quad 55x^2-145x+90 \\ \quad \underline{55x^2-55x} \\ \qquad -90x+90 \\ \qquad \underline{-90x+90} \\ \qquad\qquad 0 \end{array}$$

$35x^3+20x^2-145x+90=0 \Leftrightarrow (x-1)(35x^2+55x-90)=0 \Leftrightarrow$

$x-1=0 \vee 35x^2+55x-90=0 \Leftrightarrow x=1 \vee 7x^2+11x-18=0 \Leftrightarrow$

$x=1 \vee x=\frac{-11\pm\sqrt{11^2-4\cdot7\cdot(-18)}}{2\cdot7}=\frac{-11\pm\sqrt{625}}{14} \Leftrightarrow$

$x=1 \vee x=\frac{-11+25}{14}=1 \vee x=\frac{-11-25}{14}=-\frac{18}{7}$

$f(1)=\frac{5\cdot1}{1^2+1-6}=-\frac{5}{4}$ og $f\left(-\frac{18}{7}\right)=\frac{5\cdot\left(-\frac{18}{7}\right)}{\left(-\frac{18}{7}\right)^2+\left(-\frac{18}{7}\right)-6}=\frac{-\frac{90}{7}}{\frac{324}{49}-\frac{18}{7}-6}=\frac{105}{16}$

Skjæringspunkter i $\left(1,-\frac{5}{4}\right)$ og $\left(-\frac{18}{7},\frac{105}{16}\right)$.

e) Flatestykket ligger både over og under x-aksen, så vi må integrere over to områder:

$$F_1 = \int_{-1}^{0} \frac{5x}{x^2+x-6}\,dx - \int_{0}^{1} \frac{5x}{x^2++x-6}\,dx$$

$$\frac{5x}{x^2+x-6} = \frac{5x}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3} \quad \Leftrightarrow \quad 5x = A(x+3) + B(x-2)$$

$$x=-3 \;\Rightarrow\; 5\cdot(-3) = A(-3+3) + B(-3-2) \quad \Leftrightarrow \quad 5B = 15 \quad \Leftrightarrow \quad B = 3$$

$$x=2 \;\Rightarrow\; 5\cdot 2 = A(2+3) + B(2-2) \quad \Leftrightarrow \quad 5A = 10 \quad \Leftrightarrow \quad A = 2$$

$$F_1 = \int_{-1}^{0} \frac{5x}{x^2+x-6}\,dx - \int_{0}^{1} \frac{5x}{x^2++x-6}\,dx = \int_{-1}^{0}\left(\frac{2}{x-2}+\frac{3}{x+3}\right)dx - \int_{0}^{1}\left(\frac{2}{x-2}+\frac{3}{x+3}\right)dx$$

$$= \left[2\ln|x-2| + 3\ln|x+3|\right]_{-1}^{0} - \left[2\ln|x-2| + 3\ln|x+3|\right]_{0}^{1}$$

$$= \left(\left(2\ln|-2| + 3\ln|3|\right) - \left(2\ln|-3| + 3\ln|2|\right)\right) - \left(\left(2\ln|-1| + 3\ln|4|\right) - \left(2\ln|-2| + 3\ln|3|\right)\right)$$

$$= 2\ln 2 + 3\ln 3 - 2\ln 3 - 3\ln 2 - 3\ln 4 + 2\ln 2 + 3\ln 3 = 4\ln 3 + \ln 2 - 3\ln 2^2$$

$$= 4\ln 3 + \ln 2 - 6\ln 2 = \underline{\underline{4\ln 3 - 5\ln 2}} \approx \underline{\underline{0{,}929}}$$

f)

$$F_2 = \int_{-\frac{18}{7}}^{1}\left(-\frac{35}{16}x + \frac{15}{16}\right)dx - \int_{-\frac{18}{7}}^{1} \frac{5x}{x^2+x-6}\,dx$$

$$\int_{-\frac{18}{7}}^{1}\left(-\frac{35}{16}x + \frac{15}{16}\right)dx = \left[-\frac{35}{32}x^2 + \frac{15}{16}x\right]_{-\frac{18}{7}}^{1} = \left(-\frac{35}{32}\cdot 1^2 + \frac{15}{16}\cdot 1\right) - \left(-\frac{35}{32}\cdot\left(-\tfrac{18}{7}\right)^2 + \frac{15}{16}\cdot\left(-\tfrac{18}{7}\right)\right)$$

$$= -\frac{35}{32} + \frac{15}{16} + \frac{405}{56} + \frac{135}{56} = \frac{2125}{224}$$

$$\int_{-\frac{18}{7}}^{1} \frac{5x}{x^2+x-6}\,dx = \int_{-\frac{18}{7}}^{1}\left(\frac{2}{x-2}+\frac{3}{x+3}\right)dx = \left[2\ln|x-2| + 3\ln|x+3|\right]_{-\frac{18}{7}}^{1}$$

$$= \left(2\ln|1-2| + 3\ln|1+3|\right) - \left(2\ln\left|-\tfrac{18}{7}-2\right| + 3\ln\left|-\tfrac{18}{7}+3\right|\right)$$

$$= 2\ln|-1| + 3\ln 4 - 2\ln\left|-\tfrac{32}{7}\right| - 3\ln\left|\tfrac{3}{7}\right| = 3\ln 4 - 2(\ln 32 - \ln 7) - 3(\ln 3 - \ln 7)$$

$$= 3\ln 2^2 - 2\ln 2^5 + 2\ln 7 - 3\ln 3 + 3\ln 7 = 6\ln 2 - 10\ln 2 + 2\ln 7 - 3\ln 3 + 3\ln 7$$

$$= -4\ln 2 + 5\ln 7 - 3\ln 3$$

$$F_2 = \frac{2125}{224} - \left(-4\ln 2 + 5\ln 7 - 3\ln 3\right) = \underline{\underline{\frac{2125}{224} + 4\ln 2 - 5\ln 7 + 3\ln 3}} \approx \underline{\underline{5{,}83}}$$

Oppgave 7.63

a) $f(x) = \dfrac{\sin x}{1+\cos x}$

$$\Rightarrow \quad f'(x) = \frac{\cos x \cdot (1+\cos x) - \sin x \cdot (-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \overbrace{\cos^2 x + \sin^2 x}^{=1}}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2} = \frac{\cancel{(1+\cos x)}}{(1+\cos x)\cdot \cancel{(1+\cos x)}} = \underline{\underline{\frac{1}{1+\cos x}}}$$

b) $f'(x) = \dfrac{1}{1+\cos x} \quad \Rightarrow \quad f''(x) = \dfrac{0\cdot(1+\cos x) - 1\cdot(-\sin x)}{(1+\cos x)^2} = \dfrac{\sin x}{(1+\cos x)^2}$

Vendepunkt der $f''(x) = 0 \quad \Rightarrow \quad \dfrac{\sin x}{(1+\cos x)^2} = 0 \quad \Leftrightarrow \quad \sin x = 0 \quad \overset{x\in\langle -\pi, \pi\rangle}{\Leftrightarrow} \quad x = 0$

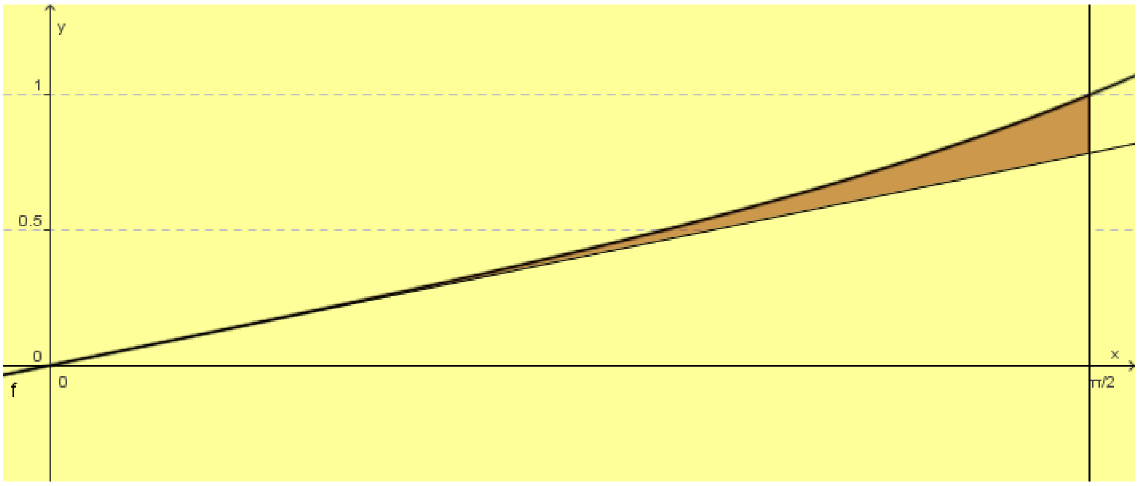
$f(0) = \dfrac{\sin 0}{1+\cos 0} = \dfrac{0}{1+1} = 0$ $\qquad$ $\underline{\underline{\text{Vendepunkt i } (0,0).}}$

c) Tangeringspunkt $(0,0)$

Tangentens stigningstall: $a = f'(0) = \dfrac{1}{1+\cos 0} = \dfrac{1}{1+1} = \dfrac{1}{2}$

$y = \dfrac{1}{2}x + b \quad \Rightarrow \quad 0 = \dfrac{1}{2}\cdot 0 + b \quad \Leftrightarrow \quad b = 0$ $\qquad$ $\underline{\underline{\text{Vendetangenten gitt ved } y = \tfrac{1}{2}x.}}$

d)



Avgrenset flatestykke gitt ved $F=\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos x}dx-\int_0^{\frac{\pi}{2}}\frac{1}{2}xdx$

Innfører $u=1+\cos x \;\Rightarrow\; \frac{du}{dx}=-\sin x \quad \left|\cdot(-1)dx\right. \;\Leftrightarrow\; (-1)du=\sin dx$

$x=0 \;\Rightarrow\; u=1+\cos 0=1+1=2$

$x=\frac{\pi}{2} \;\Rightarrow\; u=1+\cos\frac{\pi}{2}=1+0=1$

$$\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos x}\cdot\sin xdx=\int_2^1\frac{1}{u}\cdot(-1)du=\left[-\ln|u|\right]_2^1=-\ln 1-(-\ln 2)=\ln 2$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{2}xdx=\left[\frac{1}{2}\cdot\frac{1}{2}x^2\right]_0^{\frac{\pi}{2}}=\frac{1}{4}\cdot\left(\frac{\pi}{2}\right)^2-0=\frac{\pi^2}{16}$$

$$\underline{\underline{F=\ln 2-\frac{\pi^2}{16}}}$$

8.1 Differensiallikninger

Oppgave 8.10

a) $y'+2y=0 \,|\cdot e^{2x} \quad\Leftrightarrow\quad y'\cdot e^{2x}+2y\cdot e^{2x}=0\cdot e^{2x} \quad\Leftrightarrow\quad \left(y\cdot e^{2x}\right)'=0 \quad\Leftrightarrow$

$y\cdot e^{2x}=C \;\left|\cdot\frac{1}{e^{2x}}\right. \quad\Leftrightarrow\quad y=\frac{C}{e^{2x}} \quad\Leftrightarrow\quad \underline{\underline{y=Ce^{-2x}}}$

b) $y'-4y=0 \,|\cdot e^{-4x} \quad\Leftrightarrow\quad y'\cdot e^{-4x}-4y\cdot e^{-4x}=0\cdot e^{-4x} \quad\Leftrightarrow\quad \left(y\cdot e^{-4x}\right)'=0 \quad\Leftrightarrow$

$y\cdot e^{-4x}=C \;|\cdot e^{4x} \quad\Leftrightarrow\quad \underline{\underline{y=Ce^{4x}}}$

c) $y'+0,001y=0 \,|\cdot e^{0,001x} \quad\Leftrightarrow\quad y'\cdot e^{0,001x}+0,001y\cdot e^{0,001x}=0\cdot e^{0,001x} \quad\Leftrightarrow\quad \left(y\cdot e^{0,001x}\right)'=0 \quad\Leftrightarrow$

$y\cdot e^{0,001x}=C \;\left|\cdot\frac{1}{e^{0,001x}}\right. \quad\Leftrightarrow\quad y=\frac{C}{e^{0,001x}} \quad\Leftrightarrow\quad \underline{\underline{y=Ce^{-0,001x}}}$

d) $4y'+12y=0 \quad\Leftrightarrow\quad y'+3y=0 \;|\cdot e^{3x} \quad\Leftrightarrow\quad y'\cdot e^{3x}+3y\cdot e^{3x}=0\cdot e^{3x} \quad\Leftrightarrow\quad \left(y\cdot e^{3x}\right)'=0 \quad\Leftrightarrow$

$y\cdot e^{3x}=C \;\left|\cdot\frac{1}{e^{3x}}\right. \quad\Leftrightarrow\quad y=\frac{C}{e^{3x}} \quad\Leftrightarrow\quad \underline{\underline{y=Ce^{-3x}}}$

Oppgave 8.11

a) $y'+y=1 \,|\cdot e^{x} \quad\Leftrightarrow\quad y'\cdot e^{x}+y\cdot e^{x}=1\cdot e^{x} \quad\Leftrightarrow\quad \left(y\cdot e^{x}\right)'=e^{x} \quad\Leftrightarrow$

$y\cdot e^{x}=\int e^{x}dx \quad\Leftrightarrow\quad y\cdot e^{x}=e^{x}+C \;\left|\cdot\frac{1}{e^{x}}\right. \quad\Leftrightarrow\quad y=1+\frac{C}{e^{x}} \quad\Leftrightarrow\quad \underline{\underline{y=Ce^{-x}+1}}$

b) $y'-2y=6 \,|\cdot e^{-2x} \quad\Leftrightarrow\quad y'\cdot e^{-2x}-2y\cdot e^{-2x}=6\cdot e^{-2x} \quad\Leftrightarrow\quad \left(y\cdot e^{-2x}\right)'=6e^{-2x} \quad\Leftrightarrow$

$y\cdot e^{-2x}=\int 6e^{-2x}dx \quad\Leftrightarrow\quad y\cdot e^{-2x}=6\cdot\frac{1}{-2}e^{-2x}+C \;\left|\cdot\frac{1}{e^{-2x}}\right. \quad\Leftrightarrow\quad y=-3+\frac{C}{e^{-2x}} \quad\Leftrightarrow\quad \underline{\underline{y=Ce^{2x}-3}}$

c) $y'-0,002y=-0,004 \,|\cdot e^{-0,002x} \quad\Leftrightarrow\quad y'\cdot e^{-0,002x}-0,002y\cdot e^{-0,002x}=-0,004\cdot e^{-0,002x} \quad\Leftrightarrow$

$\left(y\cdot e^{-0,002x}\right)'=-0,004e^{-0,002x} \quad\Leftrightarrow\quad y\cdot e^{-0,002x}=\int -0,004e^{-0,002x}dx \quad\Leftrightarrow$

$y\cdot e^{-0,002x}=-0,004\cdot\frac{1}{-0,002}e^{-0,002x}+C \;\left|\cdot\frac{1}{e^{-0,002x}}\right. \quad\Leftrightarrow\quad y=2+\frac{C}{e^{-0,002x}} \quad\Leftrightarrow\quad \underline{\underline{y=Ce^{0,002x}+2}}$

Oppgave 8.12

a)

$$y'+3y=0 \;\Big|\cdot e^{3x} \quad\Leftrightarrow\quad y'\cdot e^{3x}+3y\cdot e^{3x}=0\cdot e^{3x} \quad\Leftrightarrow\quad \left(y\cdot e^{3x}\right)'=0 \quad\Leftrightarrow$$

$$y\cdot e^{3x}=C \;\Big|\cdot\tfrac{1}{e^{3x}} \quad\Leftrightarrow\quad y=\frac{C}{e^{3x}} \quad\Leftrightarrow\quad y=Ce^{-3x}$$

$$x=0 \;\wedge\; y=10 \quad\Rightarrow\quad Ce^{-3\cdot 0}=10 \quad\Leftrightarrow\quad C=10 \qquad \underline{\underline{y=10e^{-3x}}}$$

b)

$$y'+3y=6 \;\Big|\cdot e^{3x} \quad\Leftrightarrow\quad y'\cdot e^{3x}+3y\cdot e^{3x}=6\cdot e^{3x} \quad\Leftrightarrow\quad \left(y\cdot e^{3x}\right)'=6e^{3x} \quad\Leftrightarrow\quad y\cdot e^{3x}=\int 6e^{3x}dx \quad\Leftrightarrow$$

$$y\cdot e^{3x}=6\cdot\tfrac{1}{3}e^{3x}+C \;\Big|\cdot\tfrac{1}{e^{3x}} \quad\Leftrightarrow\quad y=2+\frac{C}{e^{3x}} \quad\Leftrightarrow\quad y=Ce^{-3x}+2$$

$$x=0 \;\wedge\; y=10 \quad\Rightarrow\quad 2+Ce^{-3\cdot 0}=10 \quad\Leftrightarrow\quad C=8 \qquad \underline{\underline{y=8e^{-3x}+2}}$$

c)

$$y'+0,03y=0,09 \;\Big|\cdot e^{0,03x} \quad\Leftrightarrow\quad y'\cdot e^{0,03x}+0,03y\cdot e^{0,03x}=0,09\cdot e^{0,03x} \quad\Leftrightarrow\quad \left(y\cdot e^{0,03x}\right)'=0,09e^{0,03x} \quad\Leftrightarrow$$

$$y\cdot e^{0,03x}=\int 0,09e^{0,03x}dx \quad\Leftrightarrow\quad y\cdot e^{0,03x}=0,09\cdot\tfrac{1}{0,03}e^{0,03x}+C \;\Big|\cdot\tfrac{1}{e^{0,03x}} \quad\Leftrightarrow\quad y=3+\frac{C}{e^{0,03x}} \quad\Leftrightarrow$$

$$y=Ce^{-0,03x}+3$$

$$x=0 \;\wedge\; y=10 \quad\Rightarrow\quad 3+Ce^{-0,03\cdot 0}=10 \quad\Leftrightarrow\quad C=7 \qquad \underline{\underline{y=7e^{-0,03x}+3}}$$

Oppgave 8.13

a)

$$y'+y=x \;\Big|\cdot e^{x} \quad\Leftrightarrow\quad y'\cdot e^{x}+y\cdot e^{x}=x\cdot e^{x} \quad\Leftrightarrow\quad \left(y\cdot e^{x}\right)'=xe^{x} \quad\Leftrightarrow\quad y\cdot e^{x}=\int \overset{u}{x}\,\overset{v'}{e^{x}}dx \quad\Leftrightarrow$$

$$y\cdot e^{x}=\overset{u}{x}\,\overset{v}{e^{x}}-\int \overset{u'}{1}\,\overset{v}{e^{x}}dx \quad\Leftrightarrow\quad y\cdot e^{x}=xe^{x}-e^{x}+C \;\Big|\cdot\tfrac{1}{e^{x}} \quad\Leftrightarrow\quad y=x-1+\frac{C}{e^{x}} \quad\Leftrightarrow\quad \underline{\underline{y=Ce^{-x}+x-1}}$$

b)

$$y'-2y=4x+2 \;\Big|\cdot e^{-2x} \quad\Leftrightarrow\quad y'\cdot e^{-2x}-2y\cdot e^{-2x}=4x\cdot e^{-2x}+2\cdot e^{-2x} \quad\Leftrightarrow$$

$$\left(y\cdot e^{-2x}\right)'=4xe^{-2x}+2e^{-2x} \quad\Leftrightarrow\quad y\cdot e^{-2x}=\int\left(4xe^{-2x}+2e^{-2x}\right)dx \quad\Leftrightarrow$$

$$y\cdot e^{-2x}=\int \overset{u}{4x}\,\overset{v'}{e^{-2x}}dx+\int 2e^{-2x}dx \quad\Leftrightarrow\quad y\cdot e^{-2x}=\overset{u}{4x}\cdot\overset{v}{\tfrac{1}{-2}e^{-2x}}-\int \overset{u'}{4}\cdot\overset{v}{\tfrac{1}{-2}e^{-2x}}dx+\int 2e^{-2x}dx$$

$$y\cdot e^{-2x}=-2xe^{-2x}+2\cdot\tfrac{1}{-2}e^{-2x}+2\cdot\tfrac{1}{-2}e^{-2x} \quad\Leftrightarrow\quad y\cdot e^{-2x}=-2xe^{-2x}-2e^{-2x}+C \;\Big|\cdot\tfrac{1}{e^{-2x}} \quad\Leftrightarrow$$

$$y=-2x-2+\frac{C}{e^{-2x}} \quad\Leftrightarrow\quad \underline{\underline{y=Ce^{2x}-2x-2}}$$

c) $y'+2y=8x^2 \quad \Big|\cdot e^{2x} \quad \Leftrightarrow \quad y'\cdot e^{2x}+2y\cdot e^{2x}=8x^2\cdot e^{2x} \quad \Leftrightarrow$

$\left(y\cdot e^{2x}\right)'=8x^2e^{2x} \quad \Leftrightarrow \quad y\cdot e^{2x}=\int \overset{u}{8x^2}\,\overset{v'}{e^{2x}}dx \quad \Leftrightarrow \quad y\cdot e^{2x}=\overset{u}{8x^2}\cdot\overset{v}{\tfrac{1}{2}e^{2x}}-\int \overset{u'}{16x}\cdot\overset{v}{\tfrac{1}{2}e^{2x}}dx$

$y\cdot e^{2x}=4x^2e^{2x}-\int \overset{w}{8x}\,\overset{v'}{e^{2x}}\,dx \quad \Leftrightarrow \quad y\cdot e^{2x}=4x^2e^{2x}-\left(\overset{w}{8x}\cdot\overset{v}{\tfrac{1}{2}e^{2x}}-\int \overset{w'}{8}\cdot\overset{v}{\tfrac{1}{2}e^{2x}}dx\right) \quad \Leftrightarrow$

$y\cdot e^{2x}=4x^2e^{2x}-4xe^{2x}+4\cdot\tfrac{1}{2}e^{2x}+C \quad \Big|\cdot\tfrac{1}{e^{2x}} \quad \Leftrightarrow \quad y=4x^2-4x+2+\dfrac{C}{e^{2x}} \quad \Leftrightarrow$

$\underline{\underline{y=Ce^{-2x}+4x^2-4x+2}}$

Oppgave 8.14

a) $y'+y=\sin x \quad \Big|\cdot e^{x} \quad \Leftrightarrow \quad y'\cdot e^{x}+y\cdot e^{x}=\sin x\cdot e^{x} \quad \Leftrightarrow$

$\left(y\cdot e^{x}\right)'=\sin x\cdot e^{x} \quad \Leftrightarrow \quad y\cdot e^{x}=\int \sin x e^{x}dx$

$\int \overset{u}{\sin x}\,\overset{v'}{e^{x}}dx=\overset{u}{\sin x}\cdot\overset{v}{e^{x}}-\int \overset{u'}{\cos x}\cdot\overset{v}{e^{x}}dx=\sin x\cdot e^{x}-\int \overset{w}{\cos x}\cdot\overset{v'}{e^{x}}dx=\sin x\cdot e^{x}-\left(\overset{w}{\cos x}\cdot\overset{v}{e^{x}}-\int \overset{w'}{-\sin x}\cdot\overset{v}{e^{x}}dx\right)$

$\Leftrightarrow \quad \int \sin x e^{x}dx=\sin x\cdot e^{x}-\cos x\cdot e^{x}-\int \sin x e^{x}dx \quad \Leftrightarrow \quad 2\int \sin x e^{x}dx=\sin x\cdot e^{x}-\cos x\cdot e^{x}+C'$

$\Leftrightarrow \quad \int \sin x e^{x}dx=\tfrac{1}{2}\sin x\cdot e^{x}-\tfrac{1}{2}\cos x\cdot e^{x}+C$

$y\cdot e^{x}=\tfrac{1}{2}\sin x\cdot e^{x}-\tfrac{1}{2}\cos x\cdot e^{x}+C \quad \Big|\cdot\tfrac{1}{e^{x}} \quad \Leftrightarrow \quad y=\tfrac{1}{2}\sin x-\tfrac{1}{2}\cos x+\dfrac{C}{e^{x}} \quad \Leftrightarrow$

$\underline{\underline{y=Ce^{-x}+\tfrac{1}{2}\sin x-\tfrac{1}{2}\cos x}}$

b) $y=Ce^{-x}+\tfrac{1}{2}\sin x-\tfrac{1}{2}\cos x \quad \Rightarrow \quad y'=Ce^{-x}\cdot(-1)+\tfrac{1}{2}\cdot\cos x-\tfrac{1}{2}\cdot(-\sin x)=-Ce^{-x}+\tfrac{1}{2}\cos x+\tfrac{1}{2}\sin x$

$y'+y=-Ce^{-x}+\tfrac{1}{2}\cos x+\tfrac{1}{2}\sin x+\left(Ce^{-x}+\tfrac{1}{2}\sin x-\tfrac{1}{2}\cos x\right)$

$=-Ce^{-x}+\tfrac{1}{2}\cos x+\tfrac{1}{2}\sin x+Ce^{-x}+\tfrac{1}{2}\sin x-\tfrac{1}{2}\cos x=\sin x$ <u>Løsningene passer i likningen.</u>

8.2 Praktisk bruk av differensiallikninger

Oppgave 8.20

a) $y'=0,03y \quad \Leftrightarrow \quad y'-0,03y=0 \;\big|\cdot e^{-0,03t} \quad \Leftrightarrow \quad y'\cdot e^{-0,03t}-0,03y\cdot e^{-0,03t}=0\cdot e^{-0,03t} \quad \Leftrightarrow$

$\left(y\cdot e^{-0,03t}\right)'=0 \quad \Leftrightarrow \quad y\cdot e^{-0,03t}=C \;\Big|\cdot\frac{1}{e^{-0,03t}} \quad \Leftrightarrow \quad y=\frac{C}{e^{-0,03t}}=Ce^{0,03t}$

Randbetingelse: $t=0 \wedge y=20 \quad \Rightarrow \quad Ce^{0,03\cdot 0}=20 \quad \Leftrightarrow \quad C=20$

Folketallet i millioner etter t år: $\underline{\underline{y=20e^{0,03t}}}$

$t=30 \quad \Rightarrow \quad y=20e^{0,03\cdot 30}\approx 49,2$ $\quad$ Folketallet etter 30 år blir 49,2 millioner.

b) Netto utflytting på 120 000 gir likningen $y'=0,03y-0,12 \quad \Leftrightarrow$

$y'-0,03y=-0,12 \;\big|\cdot e^{-0,03t} \quad \Leftrightarrow \quad y'\cdot e^{-0,03t}-0,03y\cdot e^{-0,03t}=-0,12\cdot e^{-0,03t} \quad \Leftrightarrow$

$\left(y\cdot e^{-0,03t}\right)'=-0,12e^{-0,03t} \quad \Leftrightarrow \quad y\cdot e^{-0,03t}=\int -0,12e^{-0,03t}dt \quad \Leftrightarrow$

$y\cdot e^{-0,03t}=-0,12\cdot\frac{1}{-0,03}e^{-0,03t}+C \quad \Leftrightarrow \quad y\cdot e^{-0,03t}=4e^{-0,03t}+C \;\Big|\cdot\frac{1}{e^{-0,03t}} \quad \Leftrightarrow \quad y=Ce^{0,03t}+4$

Randbetingelse: $t=0 \wedge y=20 \quad \Rightarrow \quad Ce^{0,03\cdot 0}+4=20 \quad \Leftrightarrow \quad C=16$

Folketallet i millioner etter t år: $\underline{\underline{y=16e^{0,03t}+4}}$

$t=30 \quad \Rightarrow \quad y=16e^{0,03\cdot 30}+4\approx 43,4$

Folketallet etter 30 år blir etter denne modellen 43,4 millioner.

Oppgave 8.21

a) $y'=2-\frac{10000}{10000000}y \quad \Leftrightarrow \quad y'+0,001y=2 \;\big|\cdot e^{0,001t} \quad \Leftrightarrow \quad y'\cdot e^{0,001t}+0,001y\cdot e^{0,001t}=2\cdot e^{0,001t} \quad \Leftrightarrow$

$\left(y\cdot e^{0,001t}\right)'=2e^{0,001t} \quad \Leftrightarrow \quad y\cdot e^{0,001t}=\int 2e^{0,001t}dt \quad \Leftrightarrow$

$y\cdot e^{0,001t}=2\cdot\frac{1}{0,001}e^{0,001t}+C \quad \Leftrightarrow \quad y\cdot e^{0,001t}=2000e^{0,001t}+C \;\Big|\cdot\frac{1}{e^{0,001t}} \quad \Leftrightarrow \quad y=Ce^{-0,001t}+2000$

Randbetingelse: $t=0 \wedge y=0 \quad \Rightarrow \quad Ce^{-0,001\cdot 0}+2000=0 \quad \Leftrightarrow \quad C=-2000$

Kjemikaliemengden i tonn etter t år: $\underline{\underline{y=2000-2000e^{-0,001t}}}$

b) $t \to \infty \quad \Rightarrow \quad e^{-0,001t} \to 0 \quad \Rightarrow \quad y \to 2000$ Kjemikaliemengden vil nærme seg 2000 tonn.

c) $y = 2000 - 2000e^{-0,001 \cdot 365} \approx 612$ Etter ett år vil kjemikaliemengden være 612 tonn.

d) Kjemikaliemengden i innsjøen vil etter at utslippene har stoppet, være gitt ved $y = 612e^{-0,001t}$.

$$612e^{-0,001t} = 100 \quad \Leftrightarrow \quad e^{-0,001t} = \frac{100}{612} \quad \Leftrightarrow \quad -0,001t = \ln\frac{100}{612} \quad \Leftrightarrow \quad t = \frac{\ln\frac{100}{612}}{-0,001} \approx 1812$$

$$1812 \text{ døgn} = \frac{1812}{365} \text{år} \approx 5 \text{år}$$ Det vil gå ca. 5 år før kjemikaliemengden i innsjøen er 100 tonn.

Oppgave 8.22

a) $$y' = -0,002y \quad \Leftrightarrow \quad y' + 0,002y = 0 \;\Big|\cdot e^{0,002t} \quad \Leftrightarrow \quad y' \cdot e^{0,002t} + 0,002y \cdot e^{0,002t} = 0 \cdot e^{0,002t} \quad \Leftrightarrow$$

$$\left(y \cdot e^{0,002t}\right)' = 0 \quad \Leftrightarrow \quad y \cdot e^{0,002t} = C \;\Big|\cdot \frac{1}{e^{0,002t}} \quad \Leftrightarrow \quad y = \frac{C}{e^{0,002t}} = Ce^{-0,002t}$$

Randbetingelse: $t = 0 \wedge y = 5 \quad \Rightarrow \quad Ce^{-0,002 \cdot 0} = 5 \quad \Leftrightarrow \quad C = 5$

Mengde radioaktivt stoff etter t år: $y = 5e^{-0,002t}$

$$y = 2,5 \Rightarrow \quad 5e^{-0,002t} = 2,5 \quad \Leftrightarrow \quad e^{-0,002t} = 0,5 \quad \Leftrightarrow \quad -0,002t = \ln 0,5 \quad \Leftrightarrow \quad t = \frac{\ln 0,5}{-0,002} \approx 347$$

Stoffmengden vil være halvert etter 347 år.

b) Stoffmengden vil nå være gitt ved likningen $y' = -0,002y + 0,010 \quad \Leftrightarrow$

$$y' + 0,002y = 0,01 \;\Big|\cdot e^{0,002t} \quad \Leftrightarrow \quad y' \cdot e^{0,002t} + 0,002y \cdot e^{0,002t} = 0,01 \cdot e^{0,002t} \quad \Leftrightarrow$$

$$\left(y \cdot e^{0,002t}\right)' = 0,01e^{0,002t} \quad \Leftrightarrow \quad y \cdot e^{0,002t} = \int 0,01e^{0,002t}\, dt \quad \Leftrightarrow \quad y \cdot e^{0,002t} = 0,01 \cdot \frac{1}{0,002} e^{0,002t} + C \quad \Leftrightarrow$$

$$y \cdot e^{0,002t} = 5e^{0,002t} + C \;\Big|\cdot \frac{1}{e^{0,002t}} \quad \Leftrightarrow \quad y = 5 + \frac{C}{e^{0,002t}} = 5 + Ce^{-0,002t}$$

Randbetingelse: $t = 0 \wedge y = 5 \quad \Rightarrow \quad 5 + Ce^{-0,002 \cdot 0} = 5 \quad \Leftrightarrow \quad C = 0 \quad \Rightarrow \quad y = 5$

Mengde radioaktivt stoff vil da være konstant 5 kg.

8.3 Separable differensiallikninger

Oppgave 8.30

a) $y'+2y=0 \quad \Leftrightarrow \quad y'=-2y \;\Big|\cdot\frac{1}{y} \quad \Leftrightarrow \quad \frac{1}{y}\cdot y'=-2 \quad \Leftrightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx}=-2 \quad \Leftrightarrow \quad \frac{1}{y}\cdot dy=-2dx \quad \Leftrightarrow$

$\int\frac{1}{y}dy=\int -2dx \quad \Leftrightarrow \quad \ln|y|=-2x+C' \quad \Leftrightarrow \quad |y|=e^{-2x+C'} \quad \Leftrightarrow \quad |y|=e^{-2x}\cdot e^{C'} \quad \Leftrightarrow$

$y=\pm e^{C'}\cdot e^{-2x} \quad \Leftrightarrow \quad \underline{\underline{y=Ce^{-2x}}}$

b) $y'-2y=6 \quad \Leftrightarrow \quad y'=2y+6 \quad \Leftrightarrow \quad y'=2\cdot(y+3) \;\Big|\cdot\frac{1}{y+3} \quad \Leftrightarrow \quad \frac{1}{y+3}\cdot y'=2 \quad \Leftrightarrow$

$\frac{1}{y+3}\cdot\frac{dy}{dx}=2 \quad \Leftrightarrow \quad \frac{1}{y+3}\cdot dy=2dx \quad \Leftrightarrow \quad \int\frac{1}{y+3}dy=\int 2dx \quad \Leftrightarrow \quad \ln|y+3|=2x+C' \quad \Leftrightarrow$

$|y+3|=e^{2x+C'} \quad \Leftrightarrow \quad |y+3|=e^{2x}\cdot e^{C'} \quad \Leftrightarrow \quad y+3=\pm e^{C'}\cdot e^{2x} \quad \Leftrightarrow \quad \underline{\underline{y=Ce^{2x}-3}}$

Oppgave 8.31

a) $y'+2xy=0 \quad \Leftrightarrow \quad y'=-2xy \;\Big|\cdot\frac{1}{y} \quad \Leftrightarrow \quad \frac{1}{y}\cdot y'=-2x \quad \Leftrightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx}=-2x \quad \Leftrightarrow \quad \frac{1}{y}\cdot dy=-2xdx \quad \Leftrightarrow$

$\int\frac{1}{y}dy=\int -2xdx \quad \Leftrightarrow \quad \ln|y|=-x^2+C' \quad \Leftrightarrow \quad |y|=e^{-x^2+C'} \quad \Leftrightarrow \quad |y|=e^{-x^2}\cdot e^{C'} \quad \Leftrightarrow$

$y=\pm e^{C'}\cdot e^{-x^2} \quad \Leftrightarrow \quad \underline{\underline{y=Ce^{-x^2}}}$

b) $y'+\sin x\cdot y=0 \quad \Leftrightarrow \quad y'=-\sin x\cdot y \;\Big|\cdot\frac{1}{y} \quad \Leftrightarrow \quad \frac{1}{y}\cdot y'=-\sin x \quad \Leftrightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx}=-\sin x \quad \Leftrightarrow$

$\frac{1}{y}\cdot dy=-\sin xdx \quad \Leftrightarrow \quad \int\frac{1}{y}dy=\int -\sin xdx \quad \Leftrightarrow \quad \ln|y|=\cos x+C' \quad \Leftrightarrow \quad |y|=e^{\cos x+C'} \quad \Leftrightarrow$

$|y|=e^{\cos x}\cdot e^{C'} \quad \Leftrightarrow \quad y=\pm e^{C'}\cdot e^{\cos x} \quad \Leftrightarrow \quad \underline{\underline{y=Ce^{\cos x}}}$

c) $x^2y'+y=0 \quad \Leftrightarrow \quad x^2y'=-y \;\Big|\cdot\frac{1}{x^2y} \quad \Leftrightarrow \quad \frac{1}{y}\cdot y'=-\frac{1}{x^2} \quad \Leftrightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx}=-\frac{1}{x^2} \quad \Leftrightarrow$

$\frac{1}{y}\cdot dy=-\frac{1}{x^2}dx \quad \Leftrightarrow \quad \int\frac{1}{y}dy=\int -x^{-2}dx \quad \Leftrightarrow \quad \ln|y|=-\frac{1}{-1}x^{-1}+C' \quad \Leftrightarrow \quad |y|=e^{x^{-1}+C'} \quad \Leftrightarrow$

$|y|=e^{\frac{1}{x}}\cdot e^{C'} \quad \Leftrightarrow \quad y=\pm e^{C'}\cdot e^{\frac{1}{x}} \quad \Leftrightarrow \quad \underline{\underline{y=Ce^{\frac{1}{x}}}}$

Oppgave 8.32

a) $xy'=(x+2)y \;\Big|\cdot\frac{1}{xy} \quad \Leftrightarrow \quad \frac{1}{y}\cdot y'=\frac{x+2}{x} \quad \Leftrightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx}=1+\frac{2}{x} \quad \Leftrightarrow$

$\int\frac{1}{y}dy=\int\left(1+\frac{2}{x}\right)dx \quad \Leftrightarrow \quad \ln|y|=x+2\ln|x|+C' \quad \Leftrightarrow \quad |y|=e^{x+2\ln|x|+C'} \quad \Leftrightarrow$

$|y|=e^{x}\cdot e^{\ln x^2}\cdot e^{C'} \quad \Leftrightarrow \quad y=\pm e^{C'}\cdot e^{x}\cdot x^2 \quad \Leftrightarrow \quad \underline{\underline{y=Cx^2e^{x}}}$

b)
$$4yy'-e^x=0 \quad\Leftrightarrow\quad 4yy'=e^x \quad\Leftrightarrow\quad 4y\frac{dy}{dx}=e^x \quad\Leftrightarrow\quad 4y\,dy=e^x\,dx \quad\Leftrightarrow$$
$$\int 4y\,dy=\int e^x\,dx \quad\Leftrightarrow\quad 2y^2=e^x+C' \quad\Leftrightarrow\quad y^2=\tfrac{1}{2}e^x+\tfrac{C'}{2} \quad\Leftrightarrow$$
$$y^2=\tfrac{1}{2}e^x+C \quad\Leftrightarrow\quad \underline{\underline{y=\pm\sqrt{\tfrac{1}{2}e^x+C}}}$$

c)
$$y'=\frac{6x^2}{\sqrt{y}} \quad\Leftrightarrow\quad \sqrt{y}\,y'=6x^2 \quad\Leftrightarrow\quad y^{\frac{1}{2}}\frac{dy}{dx}=6x^2 \quad\Leftrightarrow\quad y^{\frac{1}{2}}dy=6x^2dx \quad\Leftrightarrow$$
$$\int y^{\frac{1}{2}}dy=\int 6x^2dx \quad\Leftrightarrow\quad \frac{1}{\frac{3}{2}}y^{\frac{3}{2}}=2x^3+C' \quad\Leftrightarrow\quad \tfrac{2}{3}y^{\frac{3}{2}}=2x^3+C' \quad\Leftrightarrow$$
$$y^{\frac{3}{2}}=3x^3+\tfrac{3}{2}C' \quad\Leftrightarrow\quad y^{\frac{3}{2}}=3x^3+C \quad\Leftrightarrow\quad y^3=\left(3x^3+C\right)^2 \quad\Leftrightarrow\quad \underline{\underline{y=\sqrt[3]{\left(3x^3+C\right)^2}}}$$

8.4 Logistisk vekst

Oppgave 8.40

a)

$$y' = 0{,}05y \cdot \left(1 - \tfrac{y}{30}\right) \ \big|\cdot 30 \quad \Leftrightarrow \quad 30y' = 0{,}05y(30-y) \quad \Leftrightarrow \quad \frac{30y'}{y(30-y)} = 0{,}05 \quad \Leftrightarrow$$

$$\frac{30}{y(30-y)}\frac{dy}{dt} = 0{,}05 \quad \Leftrightarrow \quad \frac{30}{y(30-y)}dy = 0{,}05dt \quad \Leftrightarrow \quad \int \frac{30}{y(30-y)}dy = \int 0{,}05dt \quad \Leftrightarrow$$

$$\int \frac{30}{y(30-y)}dy = 0{,}05t + C$$

$$\frac{30}{y(30-y)} = \frac{A}{y} + \frac{B}{30-y} \quad \Leftrightarrow \quad 30 = A(30-y) + By$$

$$y = 30 \quad \Rightarrow \quad A(30-30) + B \cdot 30 = 30 \quad \Leftrightarrow \quad B = 1$$

$$y = 0 \quad \Rightarrow \quad A(30-0) + B \cdot 0 = 30 \quad \Leftrightarrow \quad A = 1$$

$$\int \frac{30}{y(30-y)}dy = \int \left(\frac{1}{y} + \frac{1}{30-y}\right)dy = \ln|y| + \frac{1}{-1} \cdot \ln|30-y|$$

$$\ln|y| - \ln|30-y| = 0{,}05t + C \quad \Leftrightarrow \quad \ln\left|\frac{y}{30-y}\right| = 0{,}05t + C \quad \Leftrightarrow \quad \left|\frac{y}{30-y}\right| = e^{0{,}05t+C} \quad \Leftrightarrow$$

$$\left|\frac{y}{30-y}\right| = e^{0{,}05t} \cdot e^{C} \quad \Leftrightarrow \quad \frac{y}{30-y} = \pm e^{C} \cdot e^{0{,}05t} \quad \Leftrightarrow \quad \frac{y}{30-y} = C \cdot e^{0{,}05t} \quad \Leftrightarrow$$

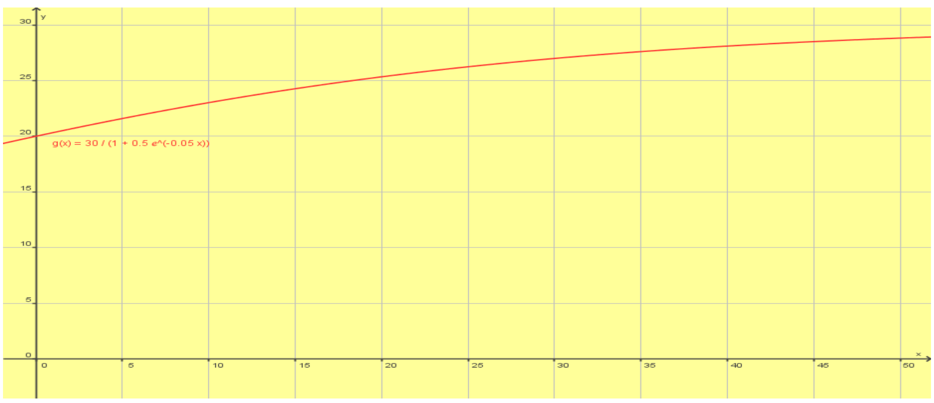
$$y = C \cdot e^{0{,}05t} \cdot (30-y) \quad \Leftrightarrow \quad \frac{1}{C \cdot e^{0{,}05t}} \cdot y = 30 - y \quad \Leftrightarrow \quad y + Ke^{-0{,}05t} \cdot y = 30 \quad \Leftrightarrow$$

$$\left(1 + Ke^{-0{,}05t}\right)y = 30 \quad \Leftrightarrow \quad y = \frac{30}{1 + Ke^{-0{,}05t}}$$

$$t = 0 \ \wedge \ y = 20 \quad \Rightarrow \quad \frac{30}{1 + Ke^{-0{,}05 \cdot 0}} = 20 \quad \Leftrightarrow \quad \frac{30}{1+K} = 20 \quad \Leftrightarrow \quad 1 + K = \frac{30}{20} \quad \Leftrightarrow \quad K = 0{,}5$$

Folketallet etter t år er gitt ved $\underline{\underline{y = \dfrac{30}{1 + 0{,}5e^{-0{,}05t}}}}$

b)



c) $y = \dfrac{30}{1+0,5e^{-0,05\cdot 10}} \approx 23,0$ Folketallet etter 10 år blir 23 millioner.

d) $\dfrac{30}{1+0,5e^{-0,05t}} = 25 \quad\Leftrightarrow\quad 1+0,5e^{-0,05t} = \frac{30}{25} \quad\Leftrightarrow\quad e^{-0,05t} = \dfrac{\frac{30}{25}-1}{0,5} = 0,4 \quad\Leftrightarrow$

$-0,05t = \ln 0,4 \quad\Leftrightarrow\quad t = \frac{\ln 0,4}{-0,05} \approx 18,3$

Etter 18,3 år vil folketallet være 25 millioner.

Oppgave 8.41

a) $y' = 0,10y\cdot\left(1-\frac{y}{200}\right) \quad |\cdot 200 \quad\Leftrightarrow\quad 200y' = 0,1y(200-y) \quad\Leftrightarrow\quad \frac{200y'}{y(200-y)} = 0,1 \quad\Leftrightarrow$

$\frac{200}{y(200-y)}\frac{dy}{dt} = 0,1 \quad\Leftrightarrow\quad \frac{200}{y(200-y)}dy = 0,1dt \quad\Leftrightarrow\quad \int\frac{200}{y(200-y)}dy = \int 0,1dt \quad\Leftrightarrow$

$\int\frac{200}{y(200-y)}dy = 0,1t + C$

$\frac{200}{y(200-y)} = \frac{A}{y} + \frac{B}{200-y} \quad\Leftrightarrow\quad 200 = A(200-y) + By$

$y = 200 \quad\Rightarrow\quad A(200-200) + B\cdot 200 = 200 \quad\Leftrightarrow\quad B = 1$

$y = 0 \quad\Rightarrow\quad A(200-0) + B\cdot 0 = 200 \quad\Leftrightarrow\quad A = 1$

$\int\frac{200}{y(200-y)}dy = \int\left(\frac{1}{y} + \frac{1}{200-y}\right)dy = \ln|y| + \frac{1}{-1}\cdot\ln|200-y|$

$\ln|y| - \ln|200-y| = 0,1t + C \quad\Leftrightarrow\quad \ln\left|\frac{y}{200-y}\right| = 0,1t + C \quad\Leftrightarrow\quad \left|\frac{y}{200-y}\right| = e^{0,1t+C} \quad\Leftrightarrow$

$\left|\frac{y}{200-y}\right| = e^{0,1t}\cdot e^{C} \quad\Leftrightarrow\quad \frac{y}{200-y} = \pm e^{C}\cdot e^{0,1t} \quad\Leftrightarrow\quad \frac{y}{200-y} = C\cdot e^{0,1t} \quad\Leftrightarrow$

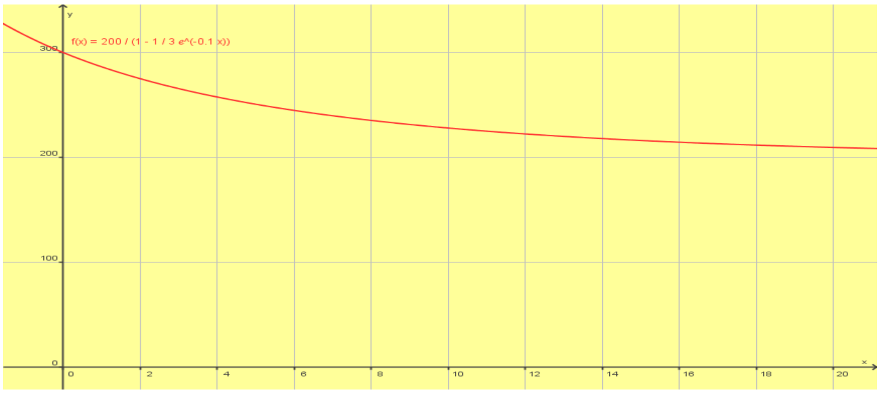
$y = C\cdot e^{0,1t}\cdot(200-y) \quad\Leftrightarrow\quad \frac{1}{C\cdot e^{0,1t}}\cdot y = 200 - y \quad\Leftrightarrow\quad y + Ke^{-0,1t}\cdot y = 200 \quad\Leftrightarrow$

$\left(1+Ke^{-0,1t}\right)y = 200 \quad\Leftrightarrow\quad y = \dfrac{200}{1+Ke^{-0,1t}}$

$t = 0 \ \wedge\ y = 300 \quad\Rightarrow\quad \dfrac{200}{1+Ke^{-0,1\cdot 0}} = 300 \quad\Leftrightarrow\quad \dfrac{200}{1+K} = 300 \quad\Leftrightarrow\quad 1+K = \frac{200}{300} \quad\Leftrightarrow\quad K = -\frac{1}{3}$

Antall rein etter t år er gitt ved $y = \dfrac{200}{1-\frac{1}{3}e^{-0,1t}}$

b)



c) $y = \dfrac{200}{1-\frac{1}{3}e^{-0{,}1\cdot 10}} \approx 228$ Antall rein etter 10 år blir 228.

d) $\dfrac{200}{1-\frac{1}{3}e^{-0{,}1t}} = 250 \quad \Leftrightarrow \quad 1-\frac{1}{3}e^{-0{,}1t} = \frac{200}{250} \quad \Leftrightarrow \quad e^{-0{,}1t} = \dfrac{1-\frac{200}{250}}{\frac{1}{3}} = 0{,}6 \quad \Leftrightarrow$

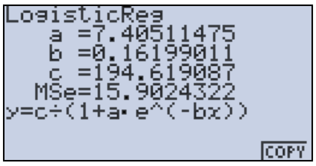
$-0{,}1t = \ln 0{,}6 \quad \Leftrightarrow \quad t = \dfrac{\ln 0{,}6}{-0{,}1} \approx 5{,}1$

Etter ca. 5 år vil antall rein være 250.

Oppgave 8.42

a)

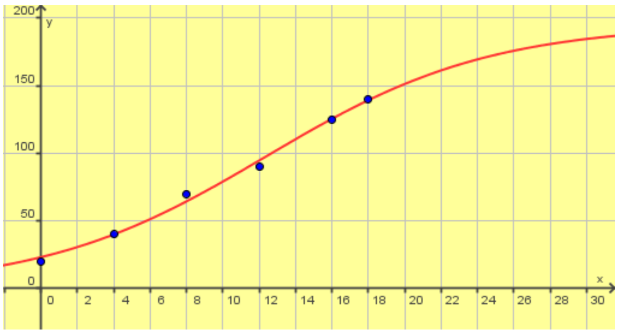
Årstall	1990	1994	1998	2002	2006	2008
t (år)	0	4	8	12	16	18
y	20	40	70	90	125	140



Antall fjellrever t år etter 1990 er gitt ved

$$y = \frac{195}{1+7{,}41e^{-0{,}162t}}$$

b)



c) $t \to \infty \quad \Rightarrow \quad e^{-0{,}162t} \to 0 \quad \Rightarrow \quad y \to 195$

Ifølge modellen vil antall fjellrever etter lang tid nærme seg 195.

d) $y = \dfrac{195}{1+7,41e^{-0,162\cdot 20}} \approx 151$ I 2010 vil antall fjellrever ifølge modellen være 151.

e) $\dfrac{195}{1+7,41e^{-0,162t}} = 175 \quad \Leftrightarrow \quad 1+7,41e^{-0,162t} = \dfrac{195}{175} \quad \Leftrightarrow \quad e^{-0,162t} = \dfrac{\frac{195}{175}-1}{7,41} \approx 0,015 \quad \Leftrightarrow$

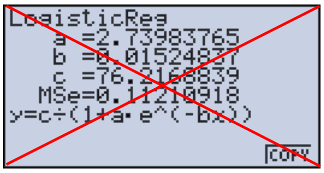
$-0,162t = \ln 0,015 \quad \Leftrightarrow \quad t = \dfrac{\ln 0,015}{-0,162} \approx 26$

Antall fjellrever vil ifølge modellen være 175 i 2016.

Oppgave 8.43

a)

Årstall	1980	1985	1990	1995	2000	2005
t (år)	0	5	10	15	20	25
y (millioner)	20	21,6	23,0	24,3	25,3	26,2

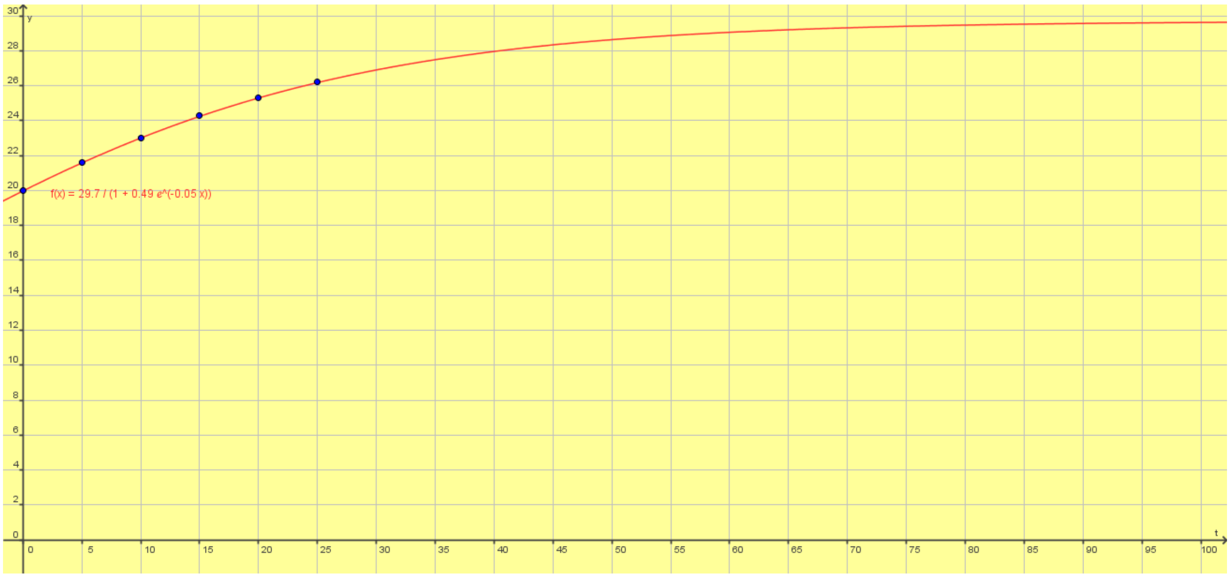


Casio gir her galt svar, - uvisst av hvilken grunn. Texas gir det riktige svaret.

Folketallet t år etter 1980 er gitt ved

$y = \dfrac{29,7}{1+0,487e^{-0,0514t}}$

b)



c) $t \to \infty \quad \Rightarrow \quad e^{-0,0514t} \to 0 \quad \Rightarrow \quad y \to 29,7$

Ifølge modellen vil folketallet etter lang tid nærme seg 29,7 millioner.

d) $y = \dfrac{29,7}{1+0,487e^{-0,0514\cdot 30}} \approx 26,9$ I 2010 vil folketallet ifølge modellen bli 26,9 millioner.

e) $\dfrac{29,7}{1+0,487e^{-0,0514t}}=25 \quad \Leftrightarrow \quad 1+0,487e^{-0,0514t}=\dfrac{29,7}{25} \quad \Leftrightarrow \quad e^{-0,0514t}=\dfrac{\frac{29,7}{25}-1}{0,487}\approx 0,386 \quad \Leftrightarrow$

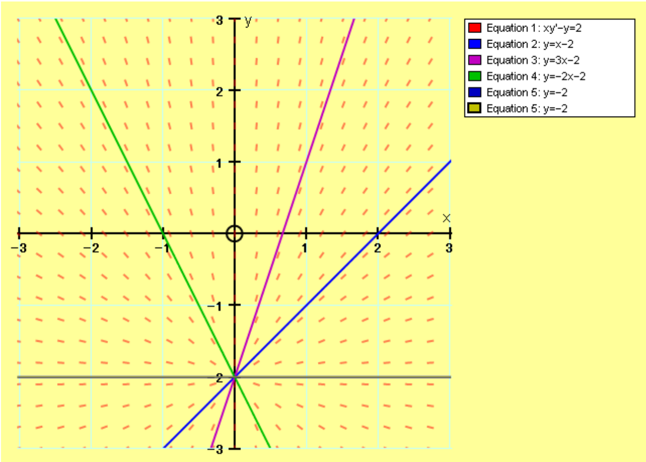
$-0,0514t=\ln 0,386 \quad \Leftrightarrow \quad t=\dfrac{\ln 0,386}{-0,0514}\approx 18,5$

I midten av 1998 vil folketallet være 25 millioner.

8.5 Retningsdiagram

Oppgave 8.50

a) + c)



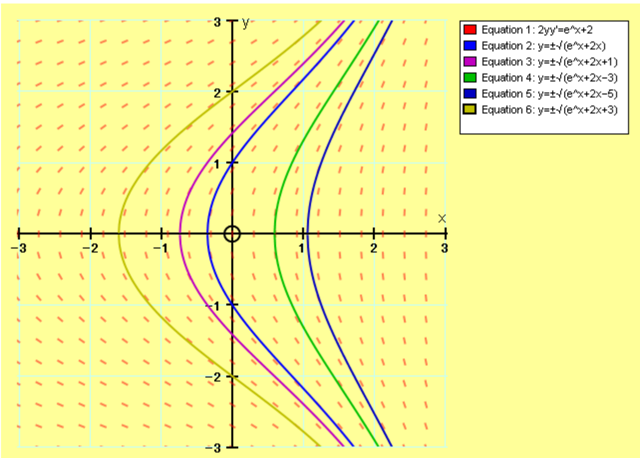
b) $xy' - y = 2 \quad \Leftrightarrow \quad xy' = y + 2 \quad \Leftrightarrow \quad y' = \frac{1}{x}(y+2) \quad \Leftrightarrow \quad \frac{1}{y+2}y' = \frac{1}{x} \quad \Leftrightarrow \quad \frac{1}{y+2}dy = \frac{1}{x}dx \quad \Leftrightarrow$

$\int \frac{1}{y+2}dy = \int \frac{1}{x}dx \quad \Leftrightarrow \quad \ln|y+2| = \ln|x| + C' \quad \Leftrightarrow \quad |y+2| = e^{\ln|x| + C'} \quad \Leftrightarrow \quad |y+2| = e^{C'} \cdot |x| \quad \Leftrightarrow$

$y + 2 = \pm e^{C'} \cdot x \quad \Leftrightarrow \quad \underline{\underline{y = Cx - 2}}$

Oppgave 8.51

a) + c)

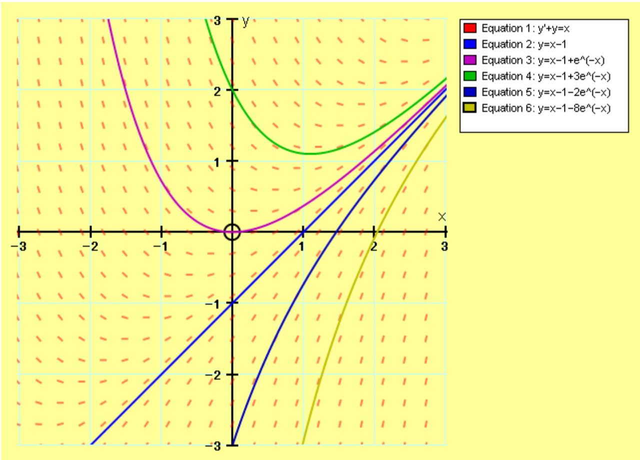


b) $2yy' = e^x + 2 \quad \Leftrightarrow \quad 2y\,dy = \left(e^x + 2\right)dx \quad \Leftrightarrow \quad \int 2y\,dy = \int \left(e^x + 2\right)dx \quad \Leftrightarrow \quad y^2 = e^x + 2x + C \quad \Leftrightarrow$

$\underline{\underline{y = \pm\sqrt{e^x + 2x + C}}}$

Oppgave 8.52

a) + c)



b) $y' + y = x \;\big|\cdot e^x \quad \Leftrightarrow \quad y'\cdot e^x + y\cdot e^x = x\cdot e^x \quad \Leftrightarrow \quad \left(y\cdot e^x\right)' = x\cdot e^x \quad \Leftrightarrow \quad y\cdot e^x = \int x\cdot e^x dx \quad \Leftrightarrow$

$y\cdot e^x = x\cdot e^x - \int 1\cdot e^x dx \quad \Leftrightarrow \quad y\cdot e^x = xe^x - e^x + C \quad \Leftrightarrow \quad y = x - 1 + \frac{C}{e^x} \quad \Leftrightarrow$

$\underline{\underline{y = x - 1 + Ce^{-x}}}$

8.6 Andreordens differensiallikninger

Oppgave 8.60

a) $y''-y'-2y=0$

Karakteristisk likning: $r^2-r-2=0 \quad \Rightarrow \quad r=\dfrac{-(-1)\pm\sqrt{(-1)^2-4\cdot1\cdot(-2)}}{2\cdot1} \quad \Leftrightarrow \quad r=-1 \;\vee\; r=2$

Generell løsning $\underline{\underline{y=Ce^{-x}+De^{2x}}}$

b) $y=Ce^{-x}+De^{2x} \quad \Rightarrow \quad y'=-Ce^{-x}+2De^{2x} \quad \Rightarrow \quad y''=Ce^{-x}+4De^{2x}$

$$\begin{aligned} y''-y'-2y &= \left(Ce^{-x}+4De^{2x}\right)-\left(-Ce^{-x}+2De^{2x}\right)-2\left(Ce^{-x}+De^{2x}\right) \\ &= Ce^{-x}+4De^{2x}+Ce^{-x}-2De^{2x}-2Ce^{-x}-2De^{2x}=0 \end{aligned}$$

$\underline{\underline{\text{Løsningen stemmer.}}}$

c) $y=Ce^{-x}+De^{2x} \quad \Rightarrow \quad y'=-Ce^{-x}+2De^{2x} \quad \Rightarrow \quad y''=Ce^{-x}+4De^{2x}$

$$\begin{aligned} y''-y'-2y &= \left(Ce^{-x}+4De^{2x}\right)-\left(-Ce^{-x}+2De^{2x}\right)-2\left(Ce^{-x}+De^{2x}\right) \\ &= Ce^{-x}+4De^{2x}+Ce^{-x}-2De^{2x}-2Ce^{-x}-2De^{2x}=0 \end{aligned}$$

$x=0 \wedge y=2 \quad \Rightarrow \quad Ce^{-0}+De^{2\cdot0}=2 \quad \Leftrightarrow \quad C+D=2 \quad \Leftrightarrow \quad C=2-D$

$x=0 \wedge y'=1 \quad \Rightarrow \quad -Ce^{-0}+2De^{2\cdot0}=1 \quad \Leftrightarrow \quad 2D-C=1$

$$\left.\begin{aligned} &2D-(2-D)=1 \;\Leftrightarrow\; 2D-2+D=1 \;\Leftrightarrow\; 3D=3 \;\Leftrightarrow\; D=1 \\ &C=2-1=1 \end{aligned}\right\} \quad \underline{\underline{y=e^{-x}+e^{2x}}}$$

Oppgave 8.61

a) $y''-7y'+12y=0$

Karakteristisk likning: $r^2-7r+12=0 \quad \Rightarrow \quad r=\dfrac{-(-7)\pm\sqrt{(-7)^2-4\cdot1\cdot12}}{2\cdot1} \quad \Leftrightarrow \quad r=3 \;\vee\; r=4$

Generell løsning $\underline{\underline{y=Ce^{3x}+De^{4x}}}$

b) $y''+y'-6y=0$

Karakteristisk likning: $r^2+r-6=0 \quad \Rightarrow \quad r=\dfrac{-1\pm\sqrt{1^2-4\cdot1\cdot(-6)}}{2\cdot1} \quad \Leftrightarrow \quad r=2 \;\vee\; r=-3$

Generell løsning $\underline{\underline{y=Ce^{2x}+De^{-3x}}}$

c) $2y''-4y=0$

Karakteristisk likning: $2r^2-4=0 \quad \Leftrightarrow \quad 2(r^2-2) \quad \Leftrightarrow \quad r=\sqrt{2} \;\vee\; r=-\sqrt{2}$

Generell løsning $\underline{\underline{y=Ce^{\sqrt{2}x}+De^{-\sqrt{2}x}}}$

d) $3y''+6y'=0$

Karakteristisk likning: $3r^2+6r=0 \quad \Leftrightarrow \quad 3r(r+2)=0 \quad \Leftrightarrow \quad r=0 \;\vee\; r=-2$

Generell løsning $y=Ce^{2\cdot 0}+De^{-2x} \quad \Leftrightarrow \quad \underline{\underline{y=C+De^{-2x}}}$

Oppgave 8.62

a) $y''-6y'+9y=0$

Karakteristisk likning: $r^2-6r+9=0 \quad \Rightarrow \quad r=\dfrac{-(-6)\pm\sqrt{(-6)^2-4\cdot 1\cdot 9}}{2\cdot 1} \quad \Leftrightarrow \quad r=3$

Generell løsning $\underline{\underline{y=(C+Dx)e^{3x}}}$

b) $y=(C+Dx)\cdot e^{3x} \quad \Rightarrow \quad y'=D\cdot e^{3x}+(C+Dx)\cdot 3De^{3x}=(D+3C+3Dx)\cdot e^{3x}$

$x=0 \wedge y=1 \;\Rightarrow\; (C+D\cdot 0)e^{3\cdot 0}=1 \;\Leftrightarrow\; C=1$

$x=0 \wedge y'=5 \;\Rightarrow\; (D+3C+3D\cdot 0)\cdot e^{3\cdot 0}=5 \;\Leftrightarrow\; 3C+D=5 \;\Rightarrow\; D=5-3\cdot 1=2$

$\underline{\underline{y=(1+2x)\cdot e^{3x}}}$

c) $y=(C+Dx)\cdot e^{3x} \quad \Rightarrow \quad y'=D\cdot e^{3x}+(C+Dx)\cdot 3De^{3x}=(D+3C+3Dx)\cdot e^{3x}$

$\Rightarrow \quad y''=3D\cdot e^{3x}+(D+3C+3Dx)\cdot 3e^{3x}=(D+D+3C+3Dx)3e^{3x}$

$$y''-6y'+9y=(D+D+3C+3Dx)3e^{3x}-6\cdot(D+3C+3Dx)\cdot e^{3x}+9\cdot(C+Dx)\cdot e^{3x}$$
$$=3e^{3x}\cdot(2D+3C+3Dx-2D-6C-6Dx+3C+3Dx)=3e^{3x}\cdot 0=0$$

<u>Løsningen stemmer.</u>

Oppgave 8.63

a) $y''+4y'+4y=0$

Karakteristisk likning: $r^2+4r+4=0 \quad \Rightarrow \quad r=\dfrac{-4\pm\sqrt{4^2-4\cdot 1\cdot 4}}{2\cdot 1} \quad \Leftrightarrow \quad r=-2$

Generell løsning $\underline{\underline{y=(C+Dx)e^{-2x}}}$

b) $y''-10y'+25y=0$

Karakteristisk likning: $r^2-10r+25=0 \quad\Rightarrow\quad r=\dfrac{-(-10)\pm\sqrt{(-10)^2-4\cdot 1\cdot 25}}{2\cdot 1} \quad\Leftrightarrow\quad r=5$

Generell løsning $\underline{\underline{y=(C+Dx)e^{5x}}}$

c) $y''-y'-12y=0$

Karakteristisk likning: $r^2-r-12=0 \quad\Rightarrow\quad r=\dfrac{-(-1)\pm\sqrt{(-1)^2-4\cdot 1\cdot(-12)}}{2\cdot 1} \quad\Leftrightarrow\quad r=-3 \ \vee\ r=4$

Generell løsning $\underline{\underline{y=Ce^{-3x}+De^{4x}}}$

Oppgave 8.64

a) $y''+9y=0$

Karakteristisk likning: $r^2+9=0 \quad\Rightarrow\quad r=\pm\sqrt{-9}=\pm\sqrt{9}\cdot\sqrt{-1}=\pm 3\sqrt{-1} \quad\Rightarrow\quad p=0 \ \wedge\ q=3$

Generell løsning $y=e^{0x}\cdot(C\sin 3x+D\cos 3x) \quad\Leftrightarrow\quad \underline{\underline{y=C\sin 3x+D\cos 3x}}$

b) $y=C\sin 3x+D\cos 3x \quad\Rightarrow\quad y'=C\cdot 3\cos 3x+D\cdot 3(-\sin 3x)=3C\cos 3x-3D\sin 3x$

$$\left.\begin{aligned} x=0 \wedge y=1 &\Rightarrow C\sin 3\cdot 0+D\cos 3\cdot 0=1 \Leftrightarrow D=1 \\ x=0 \wedge y'=1 &\Rightarrow 3C\cos 3\cdot 0-3\cdot 1\cdot\sin 3\cdot 0=1 \Leftrightarrow C=\tfrac{1}{3} \end{aligned}\right\} \underline{\underline{y=\tfrac{1}{3}\sin 3x+\cos 3x}}$$

c) $y=\frac{1}{3}\sin 3x+\cos 3x$

$\Rightarrow\quad y'=\frac{1}{3}\cdot 3\cdot\cos 3x+3\cdot(-\sin 3x)=\cos 3x-3\sin 3x$

$\Rightarrow\quad y''=3\cdot(-\sin 3x)-3\cdot 3\cdot\cos 3x=-3\sin 3x-9\cos 3x$

$y''+9y=-3\sin 3x-9\cos 3x+9\cdot\left(\frac{1}{3}\sin 3x+\cos 3x\right)=-3\sin 3x-9\cos 3x+3\sin 3x+9\cos 3x=0$

$\underline{\underline{\text{Løsningen i } b \text{ stemmer.}}}$

Oppgave 8.65

a) $y''+2y'+5y=0$

Karakteristisk likning: $r^2+2r+5=0 \quad\Rightarrow$

$$r=\frac{-2\pm\sqrt{2^2-4\cdot1\cdot5}}{2\cdot1}=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm4\cdot\sqrt{-1}}{2}=-1\pm2\sqrt{-1}\Rightarrow \quad p=-1 \quad\wedge\quad q=2$$

Generell løsning $\underline{\underline{y=e^{-x}\cdot(C\sin 2x+D\cos 2x)}}$

b) $y=e^{-x}\cdot(C\sin 2x+D\cos 2x) \quad\Rightarrow$

$$y'=-e^{-x}\cdot(C\sin 2x+D\cos 2x)+e^{-x}\cdot(2C\cos 2x+2D(-\sin 2x))$$
$$=-e^{-x}\cdot(C\sin 2x+D\cos 2x-2C\cos 2x+2D\sin 2x)=-e^{-x}\cdot((C+2D)\sin 2x+(D-2C)\cos 2x)$$

$$\left.\begin{aligned}&x=0\wedge\ y=2 \quad\Rightarrow\quad e^{-0}\cdot(C\sin 0+D\cos 0)=2 \quad\Leftrightarrow\quad D=2\\&x=0\wedge\ y'=1 \quad\Rightarrow\quad -e^{-0}\cdot((C+2\cdot2)\sin 0+(2-2C)\cos 0)=1 \quad\Leftrightarrow\quad 2C-2=1 \quad\Leftrightarrow\quad C=\tfrac{3}{2}\end{aligned}\right\}$$

$\underline{\underline{y=e^{-x}\cdot\left(\tfrac{3}{2}\sin 2x+2\cos 2x\right)}}$

c) $y=e^{-x}\cdot\left(\tfrac{3}{2}\sin 2x+2\cos 2x\right)$

$$\Rightarrow\ y'=-e^{-x}\cdot\left(\tfrac{3}{2}\sin 2x+2\cos 2x\right)+e^{-x}\cdot\left(\tfrac{3}{2}\cdot2\cdot\cos 2x+2\cdot2\cdot(-\sin 2x)\right)$$
$$=e^{-x}\cdot\left(-\tfrac{3}{2}\sin 2x-2\cos 2x+3\cos 2x-4\sin 2x\right)=e^{-x}\cdot\left(-\tfrac{11}{2}\sin 2x+\cos 2x\right)$$

$$\Rightarrow\ y''=-e^{-x}\cdot\left(-\tfrac{11}{2}\sin 2x+\cos 2x\right)+e^{-x}\cdot\left(-\tfrac{11}{2}\cdot2\cdot\cos 2x+2\cdot(-\sin 2x)\right)$$
$$=e^{-x}\cdot\left(\tfrac{11}{2}\sin 2x-\cos 2x-11\cos 2x-2\sin 2x\right)=e^{-x}\cdot\left(\tfrac{7}{2}\sin 2x-12\cos 2x\right)$$

$$y''+2y'+5y=e^{-x}\cdot\left(\tfrac{7}{2}\sin 2x-12\cos 2x\right)+2\cdot e^{-x}\cdot\left(-\tfrac{11}{2}\sin 2x+\cos 2x\right)+5\cdot e^{-x}\cdot\left(\tfrac{3}{2}\sin 2x+2\cos 2x\right)$$
$$=e^{-x}\cdot\left(\tfrac{7}{2}\sin 2x-12\cos 2x-11\sin 2x+2\cos 2x+\tfrac{15}{2}\sin 2x+10\cos 2x\right)$$
$$=e^{-x}\cdot\left(\left(\tfrac{7}{2}-11+\tfrac{15}{2}\right)\sin 2x-(12-2-10)\cos 2x\right)=e^{-x}\cdot(0\cdot\sin 2x-0\cdot\cos 2x)=0$$

Løsningen i *b* stemmer.

8.7 Udempet svingning

Oppgave 8.70

a) $0,2y''+20y=0$

Karakteristisk likning: $0,2r^2+20=0 \iff r^2=\frac{-20}{0,2}=-100 \iff r=\pm\sqrt{100}\cdot\sqrt{-1}=\pm10\sqrt{-1}$

$\Rightarrow \quad p=0 \quad \wedge \quad q=10$

Generell løsning $y=e^{0t}\cdot(C\sin10t+D\cos10t) \iff y=C\sin10t+D\cos10t$

$\Rightarrow \quad y'=10C\cdot\cos10t-10D\cdot\sin10t$

Velger positiv retning oppover $\Rightarrow$

$t=0 \wedge \overset{\text{Posisjon}}{y}=-0,10 \Rightarrow C\sin0+D\cos0=-0,10 \iff D=-0,10$

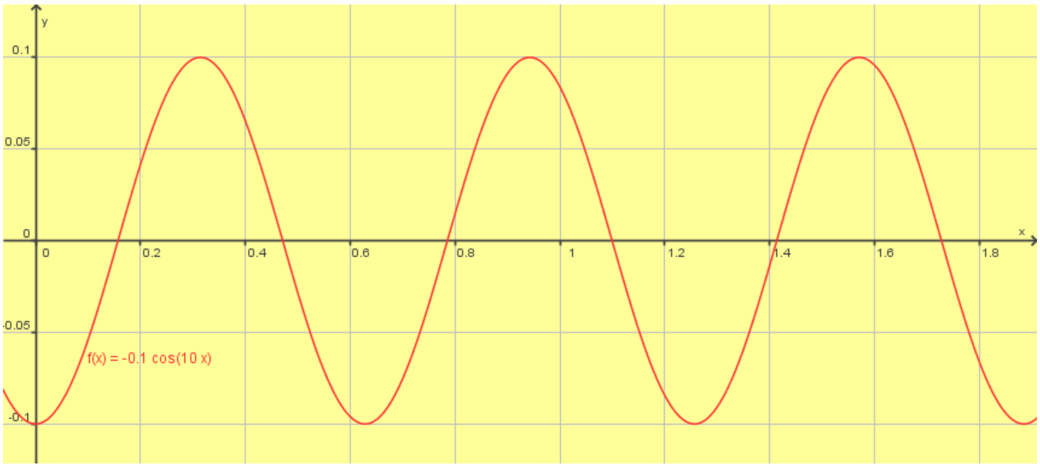
$t=0 \wedge \overset{\text{Fart}}{y'}=0 \Rightarrow 10C\cdot\cos0-10D\cdot\sin0=0 \iff C=0$

$\underline{\underline{y=-0,10\cos10t}}$

b) $A=|-0,10|=0,10$ $\quad$ Amplituden er 0,10 m.

$p=\frac{2\pi}{10}=\frac{\pi}{5}\approx0,628$ $\quad$ Perioden er 0,628 s.

c)



Oppgave 8.71

a) $0{,}15y''+21{,}6y=0$

Karakteristisk likning: $0{,}15r^2+21{,}6=0 \;\Leftrightarrow\; r^2=\frac{-21{,}6}{0{,}15}=-144 \;\Leftrightarrow$

$r=\pm\sqrt{144}\cdot\sqrt{-1}=\pm 12\sqrt{-1} \quad\Rightarrow\quad p=0 \;\wedge\; q=12$

Generell løsning $y=e^{0t}\cdot\left(C\sin 12t+D\cos 12t\right) \;\Leftrightarrow\; y=C\sin 12t+D\cos 12t$

$\Rightarrow \quad y'=12C\cdot\cos 12t-12D\cdot\sin 12t$

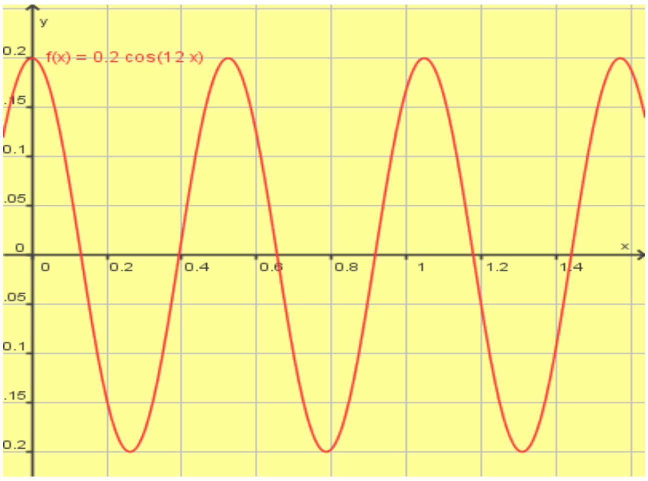
Velger positiv retning oppover $\Rightarrow$

$$\left.\begin{array}{l} t=0 \;\wedge\; \overset{\text{Posisjon}}{y}=0{,}20 \;\Rightarrow\; C\sin 0+D\cos 0=0{,}20 \;\Leftrightarrow\; D=0{,}20 \\ t=0 \;\wedge\; \overset{\text{Fart}}{y'}=0 \;\Rightarrow\; 12C\cdot\cos 0-12D\cdot\sin 0=0 \;\Leftrightarrow\; C=0 \end{array}\right\} \quad \underline{\underline{y=0{,}20\cos 12t}}$$

b) $A=|0{,}20|=0{,}20$ $\quad$ Amplituden er 0,20 m.

$p=\frac{2\pi}{12}=\frac{\pi}{6}\approx 0{,}524$ $\quad$ Perioden er 0,524 s.

c)



Oppgave 8.72

a) $0{,}1y''+10y=0$

Karakteristisk likning: $0{,}1r^2+10=0 \;\Leftrightarrow\; r^2=\frac{-10}{0{,}1}=-100 \;\Leftrightarrow$

$r=\pm\sqrt{100}\cdot\sqrt{-1}=\pm10\sqrt{-1} \quad\Rightarrow\quad p=0 \;\wedge\; q=10$

Generell løsning $y=e^{0t}\cdot\left(C\sin 10t+D\cos 10t\right) \;\Leftrightarrow\; y=C\sin 10t+D\cos 10t$

$\Rightarrow \quad y'=10C\cdot\cos 10t-10D\cdot\sin 10t$

Velger positiv retning oppover $\Rightarrow$

$$\left.\begin{array}{l} t=0 \;\wedge\; \overset{\text{Posisjon}}{y}=0{,}10 \;\Rightarrow\; C\sin 0+D\cos 0=0{,}10 \;\Leftrightarrow\; D=\frac{1}{10} \\ t=0 \;\wedge\; \overset{\text{Fart}}{y'}=\sqrt{3} \;\Rightarrow\; 10C\cdot\cos 0-10D\cdot\sin 0=\sqrt{3} \;\Leftrightarrow\; C=\frac{\sqrt{3}}{10} \end{array}\right\}\Rightarrow$$

$$y=\frac{\sqrt{3}}{10}\sin 10t+\frac{1}{10}\cos 10t$$

$$A=\sqrt{\left(\frac{\sqrt{3}}{10}\right)^2+\left(\frac{1}{10}\right)^2}=\sqrt{\frac{3}{100}+\frac{1}{100}}=\sqrt{\frac{4}{100}}=\frac{2}{10}=\frac{1}{5}\quad\Rightarrow$$

$$y=\frac{1}{5}\cdot\left(\sin 10t\cdot\frac{\sqrt{3}}{2}+\cos 10t\cdot\frac{1}{2}\right)=\frac{1}{5}\cdot\sin\left(10t+\varphi\right)$$

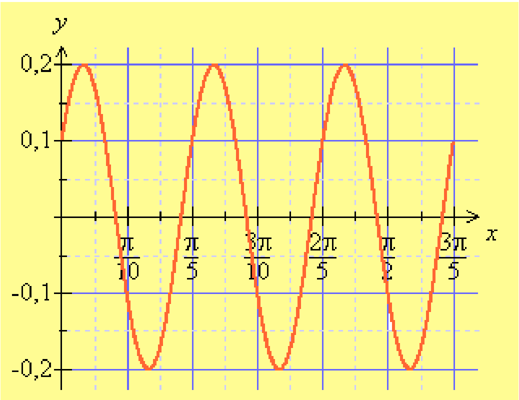
$$\cos\varphi=\frac{\sqrt{3}}{2} \;\wedge\; \sin\varphi=\frac{1}{2} \quad\Rightarrow\quad \varphi=\frac{\pi}{6}$$

$$\underline{\underline{y=\frac{1}{5}\sin\left(10t+\frac{\pi}{6}\right)=\frac{1}{5}\sin\left(10\left(t+\frac{\pi}{60}\right)\right)}}$$

b) $A=|0{,}20|=0{,}20$ $\quad$ Amplituden er 0,20 m.

$p=\frac{2\pi}{10}=\frac{\pi}{5}\approx 0{,}628$ $\quad$ Perioden er 0,628 s.

c)



Oppgave 8.73

Generellt for udempet svingning: $my''+ky=0$

Karakteristisk likning: $m\cdot r^2+k=0 \quad \Leftrightarrow \quad r^2=\frac{-k}{m} \quad \Leftrightarrow \quad r=\pm\sqrt{-\frac{k}{m}}=\pm\sqrt{\frac{k}{m}}\cdot\sqrt{-1}$

$\Rightarrow \quad y=C\sin\sqrt{\frac{k}{m}}t+D\cos\sqrt{\frac{k}{m}}t$

Svingetiden er det samme som perioden $\Rightarrow p=\frac{2\pi}{\sqrt{\frac{k}{m}}}=\frac{2\pi}{\frac{\sqrt{k}}{\sqrt{m}}}=\frac{2\pi\cdot\sqrt{m}}{\sqrt{k}}=2\pi\cdot\frac{\sqrt{m}}{\sqrt{k}}=2\pi\cdot\sqrt{\frac{m}{k}}$

$$\underline{\underline{T=2\pi\cdot\sqrt{\frac{m}{k}}}}$$

8.8 Dempet svingning

Oppgave 8.80

a) $0,5y''+3,5y'+3y=0$

Karakteristisk likning: $0,5r^2+3,5r+3=0 \quad \Rightarrow \quad r=\dfrac{-3,5\pm\sqrt{3,5^2-4\cdot 0,5\cdot 3}}{2\cdot 0,5} \quad \Leftrightarrow$

$r=-6 \;\vee\; r=-1 \quad \Rightarrow \quad y=Ce^{-6t}+De^{-t} \quad \Rightarrow \quad y'=-6Ce^{-6t}-De^{-t}$

Velger positiv retning oppover $\Rightarrow$

$$\left.\begin{array}{l} t=0 \;\wedge\; y=-0,3 \quad \Rightarrow \quad Ce^{-6\cdot 0}+De^{-0}=-0,3 \quad \Leftrightarrow \quad C+D=-0,3 \\ t=0 \;\wedge\; y'=0 \quad \Rightarrow \quad -6Ce^{-6\cdot 0}-De^{-0}=0 \quad \Leftrightarrow \quad -6C-D=0 \quad \Leftrightarrow \quad D=-6C \end{array}\right\}$$

$C+(-6C)=-0,3 \quad \Leftrightarrow \quad -5C=-0,3 \quad \Leftrightarrow \quad C=0,06 \quad \Rightarrow \quad D=-6\cdot 0,06=-0,36$

Posisjonen til loddet etter t sekunder er gitt ved $y=0,06e^{-6t}-0,36e^{-t}$.

b)

Loddet nærmer seg likevektsstillingen nedefra og etter ca.5 sekunder slutter det å svinge.

Oppgave 8.81

a) $0{,}1y''+0{,}6y'+0{,}9y=0 \quad\Leftrightarrow\quad y''+6y'+9y=0$

Karakteristisk likning: $r^2+6r+9=0 \quad\Rightarrow\quad r=\dfrac{-6\pm\sqrt{6^2-4\cdot1\cdot9}}{2\cdot1} \quad\Leftrightarrow\quad r=-3$

$\Rightarrow\quad y=(C+Dt)e^{-3t} \quad\Rightarrow\quad y'=De^{-3t}+(C+Dt)e^{-3t}\cdot(-3)=e^{-3t}\cdot(D-3C-3Dt)$

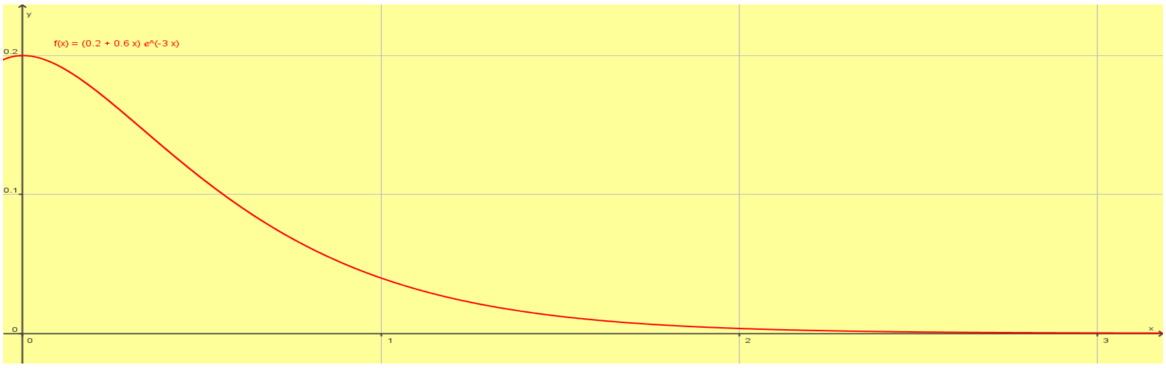
Velger positiv retning oppover $\Rightarrow$

$t=0 \wedge y=0{,}2 \quad\Rightarrow\quad (C+D\cdot0)e^0=0{,}2 \quad\Leftrightarrow\quad C=0{,}2$

$t=0 \wedge y'=0 \quad\Rightarrow\quad e^0\cdot(D-3\cdot0{,}2-3D\cdot0)=0 \quad\Leftrightarrow\quad D=0{,}6$

Posisjonen til loddet etter t sekunder er gitt ved $y=(0{,}2+0{,}6t)e^{-3t}$.

b)



Loddet nærmer seg likevektsstillingen ovenfra og etter ca.3 sekunder slutter det å svinge.

Oppgave 8.82

a) $0,4y''+0,8y'+20y=0 \quad \Leftrightarrow \quad y''+2y'+50y=0$

Karakteristisk likning: $r^2+2r+50=0 \quad \Rightarrow$

$$r=\frac{-2\pm\sqrt{2^2-4\cdot 1\cdot 50}}{2\cdot 1}=\frac{-2\pm\sqrt{196}\cdot\sqrt{-1}}{2} \quad \Leftrightarrow \quad -1\pm 7\sqrt{-1}$$

$$\Rightarrow \quad y=e^{-t}\cdot\left(C\sin 7t+D\cos 7t\right)$$

$$\Rightarrow \quad y'=-e^{-t}\cdot\left(C\sin 7t+D\cos 7t\right)+e^{-t}\cdot\left(7C\cos 7t-7D\sin 7t\right)$$

$$=e^{-t}\cdot\left(-C\sin 7t-D\cos 7t+7C\cos 7t-7D\sin 7t\right)=e^{-t}\cdot\left(\left(-C-7D\right)\sin 7t+\left(7C-D\right)\cos 7t\right)$$

Velger positiv retning oppover $\Rightarrow$

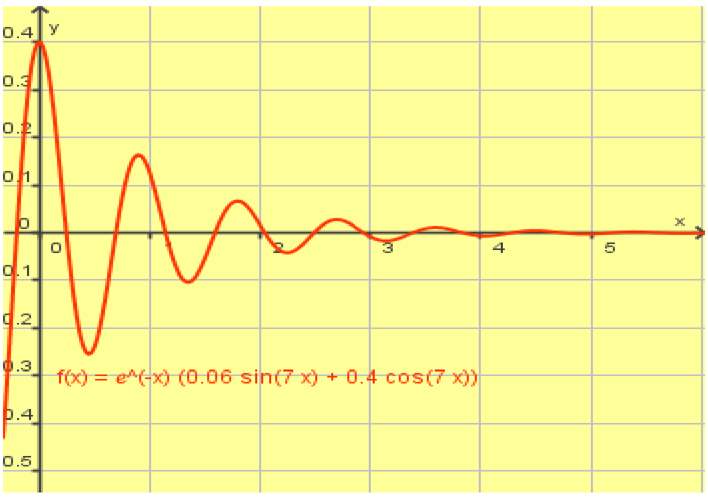
$t=0 \ \wedge \ y=0,4 \quad \Rightarrow \quad e^{-0}\cdot\left(C\sin 0+D\cos 0\right)=0,4 \quad \Leftrightarrow \quad D=0,4$

$t=0 \ \wedge \ y'=0 \quad \Rightarrow \quad e^{-0}\cdot\left(\left(-C-7\cdot 0,4\right)\sin 0+\left(7C-0,4\right)\cos 0\right)=0 \quad \Leftrightarrow$

$7C-0,4=0 \quad \Leftrightarrow \quad C\approx 0,057$

Posisjonen til loddet etter t sekunder er gitt ved $y=e^{-t}\cdot\left(0,057\sin 7t+0,4\cos 7t\right)$.

b)



Loddet svinger om likevektslinja med mindre og mindre amplitude og stopper etter ca. 6 sekunder.